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# PROCEEDINGS

OF THE

# EDINBURGH MATHEMATICAL SOCIETY.

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VOLUME III.

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SESSION 1884-85.

EDINBURGH:
1885.

PRINTED BY

JOHN LINDSAY, HIGH STREET, EDINBURGH

120433

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# PROCEEDINGS

OF THE

# EDINBURGH MATHEMATICAL SOCIETY.

## THIRD SESSION, 1884–85.

### *First Meeting, November 14th,* 1884.

THOMAS MUIR, Esq., M.A., LL.D., F.R.S.E., President, in the Chair.

For this Session the following Office-Bearers were elected :—

*President*—Mr A. J. G. BARCLAY, M.A.

*Vice-President*—Mr GEORGE THOM, M.A.

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# The Shoemaker's Knife.

By J. S. Mackay, M A., F.R.S.E.

§ 1. Some of the properties of the figure which, on account of its shape, the Greeks named the Shoemaker's Knife (ἄρβηλος) are given in the *Lemmas* attributed to Archimedes; others occur in the fourth book of Pappus's *Mathematical Collection*. The *Lemmas* (which are not extant in Greek, but have been translated from the Arabic) are generally considered to be spurious; it is, however, regarded as possible, if not probable, that the theorems among them relating to the Arbelos may be due to Archimedes. Whether they are or not, the figure and the principal proposition respecting it which Pappus gives are said by him to be "ancient." It may be added that the Arbelos does not seem to have attracted much notice from geometers, few of them having treated of it, and fewer still having added to the properties known to the ancients. (See Steiner's *Gesammelte Werke*, Vol. I., pp. 47-76, and *The Lady's and Gentleman's Diary* for 1842 and 1845).

The object of the present paper is to collect together the principal and simplest properties of the figure, and to demonstrate them in a uniform manner.

§ 2. *Figure* 1. In the arbelos AGBJCM, that is, the curvilineal figure contained by the three semicircumferences AGB, BJC, CMA, the two semicircumferences BJC, CMA are together equal to the semicircumference AGB.

For $\quad AB = AC + BC$;

and the circumferences of circles are proportional to their diameters; therefore the semicircumference AGB = semicircumference AMC + semicircumference BJC.

§ 3. *Figure* 1. The arbelos AGBJCM is equal to the circle whose diameter is CG, the common tangent at C to BJC, CMA.

For
$$AB^2 = AC^2 + BC^2 + 2\,AC \cdot BC,$$
$$= AC^2 + BC^2 + 2\,CG^2.$$

Now circles are proportional to the squares on their diameters; therefore the semicircle on AB = semicircle on AC + semicircle on BC + circle on CG;

therefore the arbelos AGBJCM = circle on CG.

Archimedes, *Lemma* 4.

§ 4. *Figure* 1. The two circles inscribed in the arbelos and touching CG are equal.

Let HJK, LMN be the two circles. Draw the diameters HP, LQ parallel to AB.

Because N is the external centre of similitude of the circles AGB, LMN, and because AB, QL are parallel diameters;
therefore the points A, Q, N, and B, L, N are collinear.
Because M is the internal centre of similitude of the circles AMC, LMN, and because AC, LQ are parallel diameters;
therefore the points A, M, L, and C, M, Q are collinear.

Let AN meet CG produced at Y, and let AL meet the semicircumference AGB at R. Join BR.

Because YC is perpendicular to AB,
and BN is perpendicular to AY;
therefore L is the orthocentre of triangle YAB;
therefore AL produced will be perpendicular to BY.
But AR is perpendicular to BR;
therefore BR and RY form one straight line.
Now since angles AMC, ARB are right,
therefore CQ is parallel to BY;

therefore $$\mathrm{AB} : \mathrm{BC} = \mathrm{AY} : \mathrm{YQ},$$
$$= \mathrm{AC} : \mathrm{QL};$$

therefore $$\mathrm{LQ} = \frac{\mathrm{AC} \cdot \mathrm{BC}}{\mathrm{AB}}.$$

Similarly $$\mathrm{HP} = \frac{\mathrm{BC} \,.\, \mathrm{AC}}{\mathrm{AB}};$$

therefore $$\mathrm{HP} = \mathrm{LQ}.$$

Archimedes, *Lemma* 5.

Cor. *Figures* 2, 3. If the circles $\mathrm{AMC}_1$, $\mathrm{BJC}_2$ intersect, or have no point in common, the circles HJK, LMN are equal, provided CG be the radical axis of $\mathrm{AMC}_1$, $\mathrm{BJC}_2$.

For it may be proved as before that
$$\mathrm{AB} : \mathrm{BC}_1 = \mathrm{AY} : \mathrm{YQ},$$
$$= \mathrm{AC} : \mathrm{QL};$$

therefore $$\mathrm{LQ} = \frac{\mathrm{AC} \cdot \mathrm{BC}_1}{\mathrm{AB}}.$$

Similarly $$\mathrm{HP} = \frac{\mathrm{BC} \cdot \mathrm{AC}_2}{\mathrm{AB}}.$$

Now $AC \cdot BC_1 = AC \cdot BC \mp AC \cdot CC_1$,
and $BC \cdot AC_2 = BC \cdot AC \mp BC \cdot CC_2$.
But $AC \cdot CC_1 = BC \cdot CC_2$, since C, being a point on the radical axis of $AMC_1$ and $BJC_2$, has equal potencies with respect to these two circles;
therefore $HP = LQ$.

This extension of the theorem of § 4, due to an Arabian mathematician, Alkauhi, is given in Borelli's *Apollonii Pergaei Conicorum Lib. V. VI. VII. et Archimedis Assumptorum Liber* (Florentiæ, 1661) pp. 393-5.

§ 5. *Figure* 1. The common tangent to the two circles AMC, LMN at M passes through B, and the common tangent to the two circles BJC, HJK at J passes through A.

For the angles ACL, ANL are right;
therefore the points A, C, L, N are concyclic;
therefore $BA \cdot BC = BN \cdot BL$,
that is, B has equal potencies with respect to the two circles AMC, LMN;
therefore B is on the radical axis of the two circles.
Now M is also on the radical axis;
therefore BM is the radical axis, or common tangent at M.

Similarly for AJ.

Cor. 1. $BM = BG$, and $AJ = AG$.
For $BM^2 = BA \cdot BC = BG^2$; and $AJ^2 = AB \cdot AC = AG^2$.
Cor. 2. BM bisects CL at V, and AJ bisects CH at U.
For the radical axis of two circles bisects their common tangents.
Cor. 3. Hence is derived a method of finding $O_2$ and $O_1$, the centres of the circles LMN, HJK.
From B draw BM tangent to the circle AMC, and cutting CG in V; make $VL = CV$; and through L draw LQ parallel to AB. If F be the centre of the circle AMC, FM produced will meet LQ in $O_2$,

Similarly for $O_1$.

Cor. 4. *Figure* 4. If AG cuts AMC at T, and BG cuts BJC at W, TW is a common tangent to AMC, BJC.

Join CT, CW, and let CG, TW intersect at I.

Since angles ATC, AGB, CWB are right,
therefore CTGW is a rectangle;
therefore IT and IW are each equal to IC;
therefore IT and IW are tangents to AMC, BJC.

§ 6. *Figure* 5. The first corollary of § 5 is a particular case of the following theorem:

Let AGB be a semicircle, and CG be perpendicular to AB. If a variable circle HJK be described to touch CG and the arc BG, and from A a tangent AJ be drawn to it, the length of AJ is constant.

Let D and O be the centres of AGB and HJK; DO will pass through K. Join OA, OH, OJ, and draw OX perpendicular to AB.

$$\begin{aligned}
\text{Then } AO^2 &= AD^2 + DO^2 + 2\ AD{\cdot}DX,\\
&= AD^2 + (AD - CX)^2 + 2\ AD\ (DC + CX),\\
&= AD^2 + AD^2 - 2\ AD{\cdot}CX + CX^2 + 2\ AD{\cdot}DC + 2AD{\cdot}CX,\\
&= 2\ AD^2 + 2\ AD{\cdot}DC + CX^2,\\
&= 2\ AD{\cdot}AC + CX^2;
\end{aligned}$$

therefore $AO^2 - CX^2 = 2\ AD{\cdot}AC$;

therefore $AJ^2 = AB{\cdot}AC$.

The theorem is still true if the variable circle touch CG produced and the arc AG externally. It is also true when CG the perpendicular to AB touches the semicircle, or falls entirely outside it (*Figure* 6), the contact in these cases being necessarily external.

Leybourn's *Mathematical Repository* (New Series), Vol VI., Part I., pp. 209-211.

§ 7. *Figure* 1. The theorem of § 4 may also be proved thus:

Let D, E, F be the centres of the semicircles AGB, BJC, CMA, and let $O_1$, $O_2$ be the centres of the circles HJK, LMN.

From $O_1$, $O_2$ draw $O_1X_1$, $O_2X_2$ perpendicular to AB; and join $DO_1$, $EO_1$, $DO_2$, $FO_2$.

Then $DO_1$ passes through K, $EO_1$ through J, $DO_2$ through N, and $FO_2$ through M.

Also $CX_1$ = the radius of HJK, $CX_2$ = the radius of LMN;

$$\begin{aligned}
AD &= \tfrac{1}{2}AB & &= \tfrac{1}{2}AC + \tfrac{1}{2}BC = FE;\\
FD &= AD - AF & &= FE - FC \quad = CE;\\
AF &= FD + DC & &= CE + DC \quad = DE.
\end{aligned}$$

Now $FO_2^2 - DO_2^2 = FX_2^2 - DX^2$.

But
$$\begin{aligned}
FO_2^2 - DO_2^2 &= (FM + MO_2)^2 - (DN - NO_2)^2,\\
&= (FC + CX_2)^2 - (AD - CX_2)^2,\\
&= FC^2 - AD^2 + 2\ AE{\cdot}CX_2;
\end{aligned}$$

and
$$\begin{aligned}
FX_2^2 - DX_2^2 &= (FC - CX_2)^2 - (DC - CX_2)^2,\\
&= FC^2 - DC^2 - 2\ FD{\cdot}CX_2;
\end{aligned}$$

therefore $FC^2 - DC^2 - 2FD\cdot CX_2 = FC^2 - AD^2 + 2\ AE\cdot CX_2$;

therefore $AD^2 - DC^2 = 2\ (AE + FD)\cdot CX_2,$
$= 2\ AB\cdot CX_2.$

Again $DO_1^2 - EO_1^2 = DX_1^2 - EX_1^2.$

But $DO_1^2 - EO_1^2 = (DK - KO_1)^2 - (EJ + JO_1)^2,$
$= (AD - CX_1)^2 - (CE + CX_1)^2,$
$= AD^2 - CE^2 - 2\ FB\cdot CX_1$;

and $DX_1^2 - EX_1^2 = (DC + CX_1)^2 - (CE - CX_1)^2,$
$= DC^2 - CE^2 + 2\ AF\cdot CX_1$;

therefore $AD^2 - CE^2 - 2\ FB\cdot CX_1 = DC^2 - CE^2 + 2\ AF\cdot CX_1$;

therefore $AD^2 - DC^2 = 2\ (AF + FB)\cdot CX_1,$
$= 2\ AB\cdot CX_1.$

Hence $2\ AB\cdot CX_1 = 2\ AB\cdot CX_2$, and $CX_1 = CX_2$.

*The Gentleman's Diary* for 1833 p. 40.

§ 8. *Figure* 1. E and $X_2$ are inverse points with respect to circle AMC; F and $X_1$ are inverse points with respect to circle BJC.

For $FE = \frac{AB}{2}$;

$$\begin{aligned} FX_2 &= FC - CX_2, \\ &= \frac{AC}{2} - \frac{AC\cdot BC}{2AB}, \\ &= \frac{AC\cdot AB - AC\cdot BC}{2\ AB}, \\ &= \frac{AC^2}{2AB}. \end{aligned}$$

Hence $FE\cdot FX_2 = \frac{AB}{2}\cdot\frac{AC^2}{2\ AB},$
$= \frac{AC^2}{4} = FC^2.$

Similarly for F and $X_1$.

Cor. 1. $EO_1 + EX_1 = BC$, and $FO_2 + FX_2 = AC$.

For $EO_1 + EX_1 = EX_2 + EX_1 = 2\ EC = BC$;
$FO_2 + FX_2 = FX_1 + FX_2 = 2\ FC = AC.$

Cor. 2. $DO_1 + DX_1 = AC$, and $DO_2 - DX_2 = BC$.

For $DO_1 = DK - O_1K$, and $DX_1 = DC + CX_1$;

therefore $DO_1 + DX_1 = DK + DC = DA + DC = AC$.

And $DO_2 = DN - O_2N$, and $DX_2 = DC - CX_2$;

therefore $DO_2 - DX_2 = DN - DC = DB - DC = BC$.

### § 9. *Figure* 1. Relations between $FO_2$ and $EO_1$.

(*a*) Their values.

$$\begin{aligned} FO_2 &= FM + O_2M, \\ &= \frac{AC}{2} + \frac{AC \cdot BC}{2AB} = \frac{AC \cdot AB + AC \cdot BC}{2AB} = \frac{AC\,(AB + BC)}{2AB}. \end{aligned}$$

Similarly $EO_1 = \dfrac{BC\,(AB + AC)}{2\,AB}$.

(*b*) Their sum.

$$\begin{aligned} FO_2 + EO_1 &= \tfrac{1}{2}\,(AC + LQ) + \tfrac{1}{2}\,(BC + HP), \\ &= \tfrac{1}{2}\,AB + HP. \end{aligned}$$

(*c*) Their difference.

$$\begin{aligned} FO_2 - EO_1 &= \tfrac{1}{2}\,(AC + LQ) - \tfrac{1}{2}\,(BC + HP), \\ &= \tfrac{1}{2}\,(AC - BC) = CD. \end{aligned}$$

(*d*) Their rectangle.

$$\begin{aligned} FO_2 \cdot EO_1 &= FX_1 \cdot EX_2, \\ &= (FC + CX_1)\,(EC + CX_2), \\ &= FC \cdot EC + FC \cdot CX_2 + EC \cdot CX_1 + CX_1 \cdot CX_2, \\ &= \frac{AC \cdot BC}{4} + \quad FE \cdot CX_1 \quad + CX_1^2, \\ &= \frac{AC \cdot BC}{4} + \frac{AC + BC}{2} \cdot \frac{AC \cdot BC}{2\,AB} + \frac{HP^2}{4}, \\ &= \frac{AC \cdot BC}{2} + \frac{HP^2}{4} = \tfrac{1}{4}\,(2CG^2 + HP^2). \end{aligned}$$

(*e*) Their ratio.

$$\begin{aligned} FO_2 : EO_1 &= \frac{AC\,(AB + BC)}{2\,AB} : \frac{BC\,(AB + AC)}{2\,AB}, \\ &= AC\,(AB + BC) : BC\,(AB + AC), \\ &= 2\,AF \cdot 2\,FB : 2\,EB \cdot 2\,AE, \\ &= AF \cdot FB : AE \cdot EB. \end{aligned}$$

### § 10. *Figure* 1. Relations between CL and CH.

(*a*) Their values.

The right-angled triangles BCV, BMF are similar;

therefore $BC : 2\,CV = BM : 2\,MF$;

therefore $BC : CL = BG : AC$;

therefore $CL = \dfrac{AC \cdot BC}{BG} = \dfrac{CG^2}{BG}$.

Similarly $CH = \dfrac{AC \cdot BC}{AG} = \dfrac{CG^2}{AG}$.

(*b*) Their sum and difference.

$$CL \pm CH = CG^2\left(\frac{1}{BG} \pm \frac{1}{AG}\right) = CG^2\left(\frac{AG \pm BG}{AG\cdot BG}\right),$$
$$= CG^2\left(\frac{AG \pm BG}{AB\cdot CG}\right) = \frac{CG}{AB}(AG \pm BG).$$

(*c*) The sum of their squares.

From the theorem, If a straight line be a common tangent to two circles which touch each other externally, that part of the tangent between the points of contact is a mean proportional between the diameters of the circles,

there results $CL^2 = AC\cdot LQ$ and $CH^2 = BC\cdot HP$;
therefore $CL^2 + CH^2 = (AC + BC)\,HP$,
$= AB\cdot HP = AC\cdot BC = CG^2$.

Cor. $AB : BC = CL^2 : LQ^2$, and $AB : AC = CH^2 : HP^2$.

For $AB : BC = AC : LQ$,
$= AC\cdot LQ : LQ^2$,
$= CL^2 : LQ^2$,
Similarly $AB : AC = CH^2 : HP^2$

Pappus, Book IV. Prop. 17.

(*d*) Their rectangle.

Since $CL^2 = AC\cdot LQ$, and $CH^2 = BC\cdot HP$;
therefore $CL^2\cdot CH^2 = AC\cdot BC\cdot HP^2 = CG^2\cdot HP^2$;
therefore $CL\cdot CH = CG\cdot HP$.

(*e*) Their ratio.

Since $CL\cdot BG = CG^2 = CH\cdot AG$;
therefore $CL : CH = AG : BG$.

§ 11. *Figure* 1. The arbelos is equal to the least circle which can be circumscribed to touch the circles HJK, LMN.

The diameter of the least circle which can be circumscribed to touch HJK, LMN will pass through $O_1$ and $O_2$, and will be equal to $O_1O_2 + HP$.

Now
$$O_1O_2^2 = PL^2,$$
$$= HL^2 + HP^2,$$
$$= (CL - CH)^2 + HP^2,$$
$$= \frac{CG^2}{AB^2}(AG - BG)^2 + \frac{CG^4}{AB^2},$$
$$= \frac{CG^2}{AB^2}\left(AG^2 + BG^2 - 2\,AG\cdot BG + CG^2\right),$$

$$=\frac{CG^2}{AB^2}(AB^2-2\,AB\cdot CG+CG^2),$$

$$=\frac{CG^2}{AB^2}(AB-CG)^2;$$

therefore $$O_1O_2=\frac{CG}{AB}(AB-CG),$$

$$=CG-\frac{CG^2}{AB},$$

$$=CG-HP;$$

therefore $O_1O_2+HP=CG$.

Leybourn's *Mathematical Repository* (New Series), Vol. VI., Part 1., pp. 155, 214.

§ 12. *Figure* 7. If from $O_1$, $O_2$, the centres of two circles which touch the semicircles AKB, AMC, and each other, there be drawn $O_1X_1$, $O_2X_2$ perpendicular to AB, and if $r_1$, $r_2$ denote the radii of circles $O_1$, $O_2$, then

$$O_1X_1+2\,r_1:2\,r_1=O_2X_2:2\,r_2.$$

Of the six centres of similitude of any three circles, every two internal centres are collinear with one external centre, and the three external centres are collinear. Hence, since J is the internal centre of similitude of the circles $O_1$ and AMC, and M is the internal centre of similitude of the circles $O_2$ and AMC, JM produced passes through E, the external centre of similitude of the circles $O_1$ and $O_2$. Since K is the external centre of similitude of the circles $O_1$ and AKB, and N the external centre of similitude of the circles $O_2$ and AKB, therefore KN produced passes also through E. Now with reference to E the external centre of similitude of circles $O_1$ and $O_2$, the points J and M are anti-homologous points, as also are K and N, and two anti-homologous points coincide at H;

therefore $EJ\cdot EM=EH^2=EK\cdot EN$.

Hence E has equal potencies with respect to the circles AMC, AKB;

therefore E is a point on the radical axis of AMC, AKB;

therefore EA is the radical axis of AMC, AKB.

Therefore $EA^2=EJ\cdot EM=EH^2$;

therefore $EA=EH$.

Now since $EA=EH$, and $O_2Q$ is parallel to EA and $=O_2H$, therefore A, Q, H are collinear.

And since $O_2Q$ and $O_1G$ are parallel and opposite in direction, therefore Q, H, G are collinear.

Let $AO_2$ meet $O_1G$ at F.

Then $$FG : O_2Q = AG : AQ,$$
$$= AX_1 : AX_2,$$
$$= O_1E : O_2E,$$
$$= r_1 : r_2.$$

But $O_2Q = r_2$; therefore $FG = r_1$, and $FO_1 = 2\ r_1$.

Lastly $$FX_1 : O_2X_2 = AX_1 : AX_2,$$
$$= r_1 : r_2;$$

therefore $$O_1X_1 + 2\ r_1 : 2\ r_1 = O_2X_2 : 2\ r_2.$$

Cor. The figure given for the theorem is susceptible of various modifications; for example, AKB may touch AMC externally at A, and then the circle $O_1$ may touch both semicircles externally or both internally. Whether AKB touch AMC internally or externally, if its centre moves off to infinity in either direction along AB, instead of the two semicircles AKB, AMC, there will be the straight line AE and the semicircle AMC, and the theorem will still be true.

Pappus, Book IV., Prop. 15.

§ 13. *Figures* 8, 9. Let the semicircles AGB, AMC touch each other internally at A, and let a series of circles $O_1$, $O_2$, $O_3$, &c., whose radii are denoted by $r_1$, $r_2$, $r_3$, &c., touch AGB, AMC and each other consecutively; if from the centres $O_1$, $O_2$, $O_3$, &c., perpendiculars $O_1X_1$, $O_2X_2$, $O_3X_3$, &c., be drawn to AB, then

(*a*) When the centre of the first circle $O_1$ lies in BC
$$\frac{O_1X_1}{r_1},\ \frac{O_2X_2}{r_2},\ \frac{O_3X_3}{r_3},\ \frac{O_4X_4}{r_4},\ \ldots\ldots\ \frac{O_nX_n}{r_n}$$
$$= 0\ ,\ 2\ ,\ 4\ ,\ 6\ ,\ \ldots\ldots\ 2\,(n-1).$$

(*b*) When the first circle touches BC
$$\frac{O_1X_1}{r_1},\ \frac{O_2X_2}{r_2},\ \frac{O_3X_3}{r_3},\ \frac{O_4X_4}{r_4},\ \ldots\ldots\ \frac{O_nX_n}{r_n}$$
$$= 1\ ,\ 3\ ,\ 5\ ,\ 7\ ,\ \ldots\ldots\ 2\,n-1.$$

In other words, the quotients obtained in the manner above described from the Shoemaker's Knife are the even numbers, those obtained from the Shoemaker's Pastehorn (as the figure AGBCM may be called) are the odd numbers.

For $$\frac{O_1X_1 + 2\ r_1}{r_1} = \frac{O_2X_2}{r_2};$$

therefore $$\frac{O_1X_1}{r_1} + 2\ = \frac{O_2X_2}{r_2}.$$

Now when the centre $O_1$ lies in BC,

$$\frac{O_1X_1}{r_1} = 0\,; \quad \text{therefore } \frac{O_2X_2}{r_2} = 2.$$

Again $\dfrac{O_2X_2}{r_2} + 2 = \dfrac{O_3X_3}{r_3}$; therefore $\dfrac{O_3X_3}{r_3} = 4$; and so on.

When the circle $O_1$ touches BC,

$$\frac{O_1X_1}{r_1} = 1\,; \quad \text{therefore } \frac{O_2X_2}{r_2} = 3,\ \frac{O_3X_3}{r_3} = 5\,; \text{ and so on.}$$

Cor. It will be seen that in figures 8 and 9 there are three circles AGB, AMC, and $O_1$ in mutual contact, and that of the series of circles $O_2$, $O_3$, $O_4$..., the first touches $O_1$, the second $O_2$, and so on, while all touch AGB, AMC. If, out of the three circles of mutual contact, instead of choosing AGB, AMC to be touched by all the series $O_2$, $O_3$, $O_4$..., we choose AGB and $O_1$, we shall have $O_2$ as before, touching AMC, and a second series $O_3$, $O_4$..., consecutively inscribed in the curvilineal space bounded by the circumferences $O_1$, $O_2$, and AGB. If we choose AMC and $O_1$ to be touched by all the series $O_2$, $O_3$, $O_4$..., we shall have $O_2$, as before, touching AGB, and a third series, $O_3$, $O_4$..., consecutively inscribed in the curvilineal space bounded by the circumferences $O_1$, $O_2$, and AMC. With respect to these two series of circles the property enunciated in § 13 holds good. It also holds good with respect to the three series of circles that may be inscribed when the semicircles AGB, AMC are replaced by straight lines perpendicular to BC at the points B and C; these straight lines being the limits towards which the two semicircles tend when their centres move off to infinity in the direction BA.

Pappus, Book IV., Props. 16, 18.

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*Second Meeting, December 12th*, 1884.

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A. J. G. Barclay, Esq., M.A., President, in the Chair.

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## Failing Cases of Fourier's Double-Integral Theorem.

### By Peter Alexander, M.A.

Fourier's Theorem, as usually stated, is

$$\phi(x) = \frac{1}{\pi}\int_0^\infty \left\{ \int_{-\infty}^{+\infty} \phi(v) \cos w(x-v)\, dv \right\} dw.$$

Most writers give this without limitation, but De Morgan (*Diff. and Int. Calc. pp.* 618 *&c.)* directs attention to what he calls the *apparent* neglect by previous writers of the limitation of the theorem to functions which satisfy the condition

$$\int_{-\infty}^{+\infty} \phi(v) dv = \text{a finite quantity.}$$

After showing that this limitation is neglected in each of three methods by which the theorem had been verified, he proceeds to show that the limitation may be removed, and that the theorem is universal. He says—

"Returning to the expression

$$\phi(x) = \frac{1}{2l}\int_{-l}^{+l} \phi(v)\, dv + \frac{1}{\pi}\Sigma \left\{ \int_{-\infty}^{+\infty} \cos w(x-v)\phi(v) dv . \Delta w \right\} \quad \text{(A.)}$$

"first observe that

$$\int (A_1x_1 + A_2x_2 + \text{\&c.})da \text{ or} \int \left[ \Sigma\ (Ax) \right] da$$

"is identical with

$$\int A_1 da.x_1 + \int A_2 da.x_2 + \text{\&c. or } \Sigma \left( \int A\ da.x \right)$$

"provided only that $x_1$, $x_2$, &c., are independent of $a$.

"Write the expression (A) in the form

$$\frac{1}{\pi}\int_{-l}^{+l} \Big\{ \tfrac{1}{2}\Delta w \cos 0\ (x-v) + \Delta w \cos \Delta w\ (x-v)$$

$$+ \Delta w \cos 2\Delta w\ (x-v) + \text{\&c.} \Big\} \phi(v)\ dv = \phi(x).$$

"This expression is absolutely true for $-l(x)l$, whatever the values of $l$ may be, and the series it contains is the limit of a set of convergent series made by diminishing $\kappa$ without limit in

$$\tfrac{1}{2}\,\Delta w \cos 0\,(x-v).\epsilon^{-o.\kappa\Delta w} + \Delta w \cos \Delta w\,(x-v).\epsilon^{-\kappa\Delta w} + \Delta w \cos 2\Delta w\,(x-v).\epsilon^{-2\kappa\Delta w} + \ldots\ldots$$

"Let $\kappa$ have any positive value, however small, and let the preceding be multiplied by $\phi(v)$ and integrated with respect to $v$ from $v = -l$ to $v = +l$; that is from $v = -(\pi \div \Delta w)$ to $v = +(\pi \div \Delta w)$; and if $x$ lie between these limits the result will be as near as we please to $\phi(x)$ if $\kappa$ be taken small enough. Since the series is convergent this might be verified by actual arithmetical operation.

"*Now, since the individual terms of the preceding diminish without limit with $\Delta w$, any one or more of them, in fact any finite and fixed number of them, might be erased or altered in any finite ratio without affecting the result.*

"If then in the first term we change $\tfrac{1}{2}$ into 1, (or if we erased the first term altogether,) the limit of the result, when $\Delta w$ is dimished without limit, is strictly

$$\frac{1}{\pi}\int_{-\pi\div 0}^{+\pi\div 0}\int_{0}^{\infty} \cos w\,(x-v).\epsilon^{-\kappa w}\,\phi\,(v)\;dv.dw = \phi\,(x) + a$$

"where $a$ and $\kappa$ are comminuent.

"Diminish $\kappa$ without limit and we have Fourier's theorem as given."

This reasoning seems plausible, but is fallacious. The fallacy is contained in the words "Now, since……the result," which I have italicised. He takes it for granted that, because the term

$$\tfrac{1}{2}\,\Delta w \cos 0\,(x-v).\epsilon^{-0.\kappa\Delta w}.\phi(v) \quad \text{or} \quad \tfrac{1}{2}\,\Delta w\;\phi(v)$$

diminishes without limit with $\Delta w$, its integral

$$\int_{\frac{-\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \tfrac{1}{2}\,\Delta w.\phi(v)\;dv$$

also dininishes without limit with $\Delta w$, which does not of necessity follow. In fact he admits that it does not in a former part of his discussion of this theorem, where he says :—"$\int \phi(v)dv \div 2l$ may in such case (*i.e.*, if $\int \phi(v)dv$ increase without limit with $l$) either increase without limit, have a finite limit, or diminish without limit." [Here $l = \dfrac{\pi}{\Delta w}$ and $\pi$ is dropped.]

To prove that $\int_{-\frac{\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \frac{1}{2}\,\Delta w.\phi(v)dv$ is not necessarily zero,

let $\phi(v) = \frac{d}{dv}.\psi(v)$. Then

$$\int_{-\frac{\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \frac{1}{2}\,\Delta w.\phi(v)dv = \frac{1}{2}\,\Delta w\left\{\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)\right\}$$

or $$\frac{1}{2}\,\Delta w\left\{\left[\psi\left(\frac{\pi}{\Delta w}\right) - \psi(v)\right] + \left[\psi(v) - \psi\left(-\frac{\pi}{\Delta w}\right)\right]\right\}$$

Now if $\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)$ become zero or remain finite when $\Delta w$ is dimished without limit the integral will become zero. But if $\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)$ increase without limit when $\Delta w$ is diminished without limit the integral will not necessarily become zero.

For example, suppose

$\phi(v) = p$ for positive values of $v$

and $= q$ „ negative „ „

Then $\psi(v) = pv$ for positive values of $v$

and $= qv$ „ negative „ „

Then $$\int_{-\frac{\pi}{\Delta w}}^{\frac{\pi}{\Delta w}} \frac{1}{2}\,\Delta w\phi(v)\,dv = \frac{1}{2}\,\Delta w\left\{\psi\left(\frac{\pi}{\Delta w}\right) - \psi\left(-\frac{\pi}{\Delta w}\right)\right\}$$

$$= \frac{1}{2}\,\Delta w\left\{p\,\frac{\pi}{\Delta w} - q\left(-\frac{\pi}{\Delta w}\right)\right\}.$$

$$= \frac{p+q}{2}\pi$$

which is not zero unless $q = -p$.

In cases such as this Fourier's Theorem fails, and instead of it we should use

$$\phi(x) = \mathop{\text{L}}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right] + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{+\infty}\phi(v)\cos w(x-v)dvdw. \qquad \text{(B)}$$

which is universal.

This follows at once from (A) which is true for all periodic functions of period $-l(x)l$.

Now a non-periodic function may be looked upon as a periodic function of infinite period $-\infty\ (x)\ \infty$. But (A) is true however great the period $-l(x)l$ be, hence its limiting form, (B), will be true for a function of infinite range, that is for a non-periodic function.

I shall now give one or two illustrations of the failure of Fourier's theorem, and show that the modified theorem (B) gives the true expression.

First example—

Let $\phi\ (x) = p$ for all positive values of $x$.
and $\qquad = q$ " negative " " .

Then $\displaystyle \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{+\infty} \phi\ (v).\cos w\ (x-v)\ dv.dw$

$$= \frac{1}{\pi}\int_0^\infty \left\{ p \int_0^\infty \cos w\ (x-v)\ dv + q \int_{-\infty}^0 \cos w\ (\chi - v)\ dv \right\}$$

$$= \frac{1}{\pi}\int_0^\infty \left\{ p \left(\frac{\sin wx}{w}\right) + q \left(-\frac{\sin wx}{w}\right)\right\} dw$$

$$= \frac{p-q}{\pi}\int_0 \frac{\sin w\, x}{w} dw$$

$$= \frac{p-q}{\pi}\left(\pm\frac{\pi}{2}\right) = \pm\frac{p-q}{2}$$

according as $x$ is positive or negative.

This is certainly not $= \phi(x)$. Hence the ordinary form of Fourier's theorem fails in this case.

But let us see what (B) gives.

$$\mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right] = \mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\left\{\int_0^l p\,dv + \int_{-l}^0 q\,dv\right\}\right]$$

$$= \mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\left\{p\,l + q\,l\right\}\right]$$

$$= \frac{p+q}{2};$$

consequently

$$\mathop{\mathrm{L}}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi\ (v)\ dv\right] + \frac{1}{\pi}\int_0^\infty \int_{-\infty}^{+\infty} \phi\ (v).\cos w\ (x-v)\ dv.dw)$$

$$= \frac{p+q}{2} \pm \frac{p-q}{2} = p \text{ or } q,$$

according as $x$ is positive or negative.

Therefore (B) holds in this case.

Second example—

Let $\phi(x) = a(1+\epsilon^{-mx})$ for positive values of $x$.

and $\qquad = b(1+\epsilon^{mx})$ ,, negative ,, ,, .

Then

$$\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}\phi(v)\cos w(x-v)\,dv.dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{\int_0^\infty a(1+\epsilon^{-mv})\cos w(x-v)\,dv\right.$$
$$\left.+\int_{-\infty}^0 b(1+\epsilon^{mv})\cos w(x-v)\,dv\right\}dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{a\left[\int_0^\infty\cos w(x-v)\,dv+\int_0^\infty\epsilon^{-mv}\cos w(x-v)\,dv\right]\right.$$
$$\left.+b\left[\int_{-\infty}^0\cos w(x-v)\,dv+\int_{-\infty}^0\epsilon^{mv}\cos w(x-v)\,dv\right\}dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{a\left[\cos wx\left(\int_0^\infty\cos wv\,dv+\int_0^\infty\epsilon^{-mv}\cos wv\,dv\right)\right.\right.$$
$$+\sin wx\left(\int_0^\infty\sin wv\,dv+\int_0^\infty\epsilon^{-mv}\sin wv\,dv\right)\Bigg]$$
$$+b\Bigg[\cos wx\left(\int_{-\infty}^0\cos wv\,dv+\int_{-\infty}^0\epsilon^{mv}\cos wv\,dv\right)$$
$$\left.\left.+\sin wx\left(\int_{-\infty}^0\sin wv\,dv+\int_{-\infty}^0\epsilon^{mv}\sin wv\,dv\right)\right]\right\}dw$$

$$=\frac{1}{\pi}\int_0^\infty\left\{a\left[\cos wx\left(0+\frac{m}{m^2+w^2}\right)+\sin wx\left(\frac{1}{w}+\frac{w}{m^2+w^2}\right)\right]\right.$$
$$\left.+b\left[\cos wx\left[0+\frac{m}{m^2+w^2}\right)+\sin wx\left(-\frac{1}{w}-\frac{w}{m^2+w^2}\right)\right]\right\}dw$$

$$=\frac{1}{\pi}\left\{(a+b)\,m\int_0^\infty\frac{\cos wx}{m^2+w^2}\,dw+(a-b)\int_0^\infty\frac{\sin wx}{w}\,dw\right.$$
$$\left.+(a-b)\int_0^\infty\frac{w\sin wx}{m^2+w^2}\,dw\right\}$$

$$=\frac{1}{\pi}\left\{(a+b)\,m.\frac{\pi}{2m}\,\epsilon^{\pm mx}+(a-b).\left(\pm\frac{\pi}{2}\right)\right.$$
$$\left.+(a-b)\left(\pm\frac{\pi}{2}\epsilon^{\mp mx}\right)\right\}$$

$$=\pm\frac{a-b}{2}+\left(\frac{a+b}{2}\pm\frac{a-b}{2}\right)\epsilon^{\mp mx}$$

the upper or lower sign according as $x$ is positive or negative.

This becomes

$$\frac{a-b}{2}+a\epsilon^{-mx} \text{ when } x \text{ is positive,}$$

and $$-\frac{a-b}{2}+b\epsilon^{-mx} \text{ when } x \text{ is negative,}$$

and is therefore not $=\phi(x)$. Hence Fourier's theorem again fails.

Let us now examine what (B) gives in this case.

$$\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right]=\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\left\{\int_0^l a\,(1+\epsilon^{-mv})\,dv + \int_{-l}^{0} b\,(1+\epsilon^{mv})\,dv\right\}\right]$$

$$=\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\left\{a\left(l+\frac{1-\epsilon^{-ml}}{m}\right)+b\left(l+\frac{1-\epsilon^{-ml}}{m}\right)\right\}\right]$$

$$=\frac{a+b}{2}.$$

Hence

$$\mathrm{L}_{l=\infty}\left[\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv\right]+\frac{1}{\pi}\int_0^\infty\int_{-\infty}^{+\infty}\phi(v)\cos w\,(x-v)\,dv.dw$$

$$=\frac{a+b}{2}\pm\left\{\frac{a-b}{2}\pm\left(\frac{a+b}{2}\pm\frac{a-b}{2}\right)\epsilon^{\pm mx}\right\}$$

$$=\left(\frac{a+b}{2}\pm\frac{a-b}{2}\right)\left(1+\epsilon^{\mp mx}\right).$$

according as $x$ is positive or negative.

This gives $a\,(1+\epsilon^{-mx})$ when $x$ is positive.

and $b\,(1+\epsilon^{mx})$ „ „ negative.

that is, it gives $\phi(x)$.

Hence (B) holds for this case.

Some writers, as Duhamel and Price, to avoid difficulty, assume that $\phi(x)$ never $=\infty$, and $\phi(x)=0$ when $x=\pm\infty$, thus ignoring all such functions as the two dealt with above.

Similarly, Todhunter annexes to the theorem the condition that

$$\frac{1}{2l}\int_{-l}^{l}\phi(v)\,dv \text{ vanishes with } \frac{1}{l}.$$

Freeman, Fourier's translator, (*Theory of Heat*, p. 351) says that Poisson first gave a direct proof that

$$\phi(x)=\left\{\frac{1}{\pi}\int_0^\infty dq\int_{-\infty}^{+\infty}d\alpha.\epsilon^{-\kappa q}\cos(qx-q\alpha)\,\phi(\alpha)\right\}_{\kappa=0}$$

$\kappa$ being put = 0 after the integrations, and that Boole regarded Fourier's theorem as unproved, unless equivalent to this. But this is just what De Morgan tried to prove, and failed in doing, as I have shown. Moreover, if we test this on the first of the two examples I have chosen, we have

$$\frac{1}{\pi}\int_0^\infty dw\int_{-\infty}^{+\infty} dv.\epsilon^{-\kappa w}\cos w\,(x-v)\,.\,\phi\,(v)$$

$$=\frac{1}{\pi}\int_0^\infty dw.\epsilon^{-\kappa w}\left\{p\int_0^\infty \cos w\,(x-v)\,dv+q\int_{-\infty}^0 \cos w\,(x-v)\,dv\right\}$$

$$=\frac{1}{\pi}\int_0^\infty dw.\epsilon^{-\kappa w}\left\{p\left(\frac{\sin wx}{w}\right)+q\left(-\frac{\sin wx}{w}\right)\right\}$$

$$=\frac{p-q}{\pi}\int_0^\infty \frac{\epsilon^{-\kappa w}\sin wx}{w}\,dw$$

$$=\frac{p-q}{\pi}\tan^{-1}\frac{x}{\kappa}.$$

If now we put $\kappa=0$ this becomes

$$\frac{p-q}{\pi}\left(\pm\frac{\pi}{2}\right)=\pm\frac{p-q}{2}$$

which is not $=\phi(x)$.

Therefore Poisson's formula gives the same result as Fourier's and seems subject to the same limitations.

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*Note.*—In the discussion which followed, Professor Chrystal and Dr Muir mentioned one or two papers* worth reading on this subject; Dr Muir also stated that he had some little hesitation in accepting the remarks on Poisson's form, and hoped Mr Alexander would examine further into the matter.

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* *G(regory?).* On Fourier's Theorem. *Cambridge Mathematical Journal,* III., pp. 288-290.

*Walton, W.* A demonstration of Fourier's Theorem. *Quarterly Journal of Mathematics,* VIII., pp. 136-138.

*Glaisher, J. W. L.* On Fourier's double-integral Theorem. *Messenger of Mathematics,* II., pp. 20-24.

*Du Bois Reymond, P.* Sur les formules de représentation des fonctions. *Comptes rendus*...Paris, XCII., pp. 915-918, 962-964.

These are in addition to the list given by Freeman at the place above cited.

# Note on a Function of two Integral Arguments.

By Thomas Muir, LL.D.

1. In the part just issued of the *American Journal of Mathematics* (vii, pp. 3, 4) Professor Cayley has occasion to deal with the function

$$\left\{i, j\right\} \equiv \left[i\right]^j + \frac{j}{1}\left[i\right]^{j-1} + \frac{j(j-1)}{1.2}\left[i\right]^{j-2} + \ldots + 1$$

where $\left[i\right]^j$ stands for $i\,(i-1)\ldots(i-j+1)$. He points out that

$$\left\{i, j\right\} = \left\{j, i\right\},$$

expresses $\{i,\ 1\}$, $\{i,\ 2\}$, ... $\{i,\ 5\}$ each in descending powers of $i$, and tabulates the function from $\{1,\ 1\}$ to $\{5,\ 5\}$.

2. The series which defines the function may be looked on as a sum of terms, each of which is a product of a number of combinations $(C_{m,r})$ and a number of permutations $(P_{n,m-r})$: that is to say, we may write the definition as follows—

$$\left\{m,\ n\right\} \equiv P_{m,n} + C_{n,1}P_{m,n-1} + C_{n,2}P_{m,n-2} + \ldots\ldots + 1.$$

Now, multiplying both sides by $m+1$, and bearing in mind that of course

$$(m+1)P_{m,n-r} = P_{m+1,n-r+1},$$

we have

$$(m+1)\left\{m,\ n\right\} = P_{m+1,n+1} + C_{n,1}P_{m+1,n} + C_{n,2}P_{m+1,n-1} + \ldots\ldots + (m+1).$$

But by definition

$$\left\{m+1, n+1\right\} = P_{m+1,n+1} + C_{n+1,1}P_{m+1,n} + C_{n+1,2}P_{m+1,n-1} + \ldots\ldots + 1,$$

and therefore by subtraction and using the fact that

$$C_{n+1,r} - C_{n,r} = C_{n,r-1}$$

we have

$$\left\{m+1, n+1\right\} - (m+1)\left\{m, n\right\} = P_{m+1,n} + C_{n,1}P_{m+1,n-1} + \ldots + 1$$
$$= \left\{m+1,\ n\right\}:$$

and hence, as our law of recurrence,

$$\left\{m+1,\ n+1\right\} = (m+1)\left\{m,\ n\right\} + \left\{m+1,\ n\right\}. \qquad \text{(I.)}$$

3. Now, since $\left\{m,\ 1\right\} = m + 1$ the first line of our table is

$$2,\ 3,\ 4,\ 5,\ 6,\ 7,\ \ldots;$$

then from (I) $2{\cdot}2 + 3$, $3{\cdot}3 + 4$, $4{\cdot}4 + 5$, $5{\cdot}5 + 6$, ... give us the items of the second line, viz.

$$7,\ 13,\ 21,\ 31,\ 43,\ \ldots;$$

similarly $3{\cdot}7 + 13$, $4{\cdot}13 + 21$, $5{\cdot}21 + 31$, ... give us the items of the third line, viz.

$$34,\ 73,\ 136,\ 229,\ \ldots;$$

and so on; the table thus being

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | | 7 | 13 | 21 | 31 | 43 |
| 3 | | | 34 | 73 | 136 | 229 |
| 4 | | | | 209 | 501 | 1045 |
| 5 | | | | | 1546 | 4051 |
| 6 | | | | | | 13327 |

4. Further, the following properties of the function may be noted; several of them, especially the last three, may be used as a check upon the preceding process of calculation.

Since $\left\{m,\ n\right\} = m\left\{m-1,\ n-1\right\} + \left\{m,\ n-1\right\}$

and $\left\{n,\ m\right\} = n\left\{n-1,\ m-1\right\} + \left\{n,\ m-1\right\}$,

therefore by subtraction and division

$$\frac{\left\{n,\ m-1\right\} - \left\{m,\ n-1\right\}}{m-n} = \left\{m-1,\ n-1\right\}. \qquad \text{(II.)}$$

Similarly we have

$$\frac{m\left\{n,\ m-1\right\} - n\left\{m,\ n-1\right\}}{m-n} = \left\{m,\ n\right\}. \qquad \text{(III.)}$$

And consequently from these two

$$m\left\{m-1,\ n\right\} + \left\{m+1,\ n\right\} = n\left\{n-1,\ m\right\} + \left\{n+1,\ m\right\}; \text{(IV.)}$$

that is to say, $m\left\{m-1,\ n\right\}+\left\{m+1,\ n\right\}$ is symmetric with respect to $m$ and $n$, a fact which may also be seen from noting that either side of (IV) is equal to

$$\left\{m,\ n+1\right\}-\left\{m,\ n\right\}+\left\{m+1,\ n\right\}.$$

Again, since from (I) we have

$$\left\{n+2, n\right\}=\left\{n+2, n+1\right\}-(n+2)\left\{n+1, n\right\}$$

$$\left\{n+1, n+2\right\}=(n+1)\left\{n, n+1\right\}+\left\{n+1, n+1\right\}$$

and $$\left\{n+1, n+1\right\}=(n+1)\left\{n, n\right\}+\left\{n+1, n\right\},$$

therefore by addition

$$\left\{n+2, n\right\}=(n+1)\left\{n, n\right\}: \qquad \text{(V.)}$$

that is to say, *the numbers in the third diagonal of the above table are known multiples of the corresponding numbers in the first diagonal.*

Again from (II) by writing $2n$ for $m$, and $n+1$ for $n$, we have

$$\left\{n+1, 2n-1\right\}-\left\{2n, n\right\}=(n-1)\left\{2n-1, n\right\}$$

and from (I)

$$\left\{n, 2n\right\}=n\left\{n-1, 2n-1\right\}+\left\{n, 2n-1\right\}$$

therefore by addition

$$\left\{2n-1, n+1\right\}=n\left[\left\{2n-1, n\right\}+\left\{2n-1, n-1\right\}\right] \text{(VI.)}$$

that is to say, *in every odd column of the table there is one item which is a known multiple of the sum of the preceding pair of items.*

Also, *the first item of the said pair is the arithmetic mean of the item immediately to the right and the item immediately above the latter.* (VII.)

This follows from (III) by putting $m=2n$ and dividing by $n$.

5. Lastly, it may be remarked that the function $\{m,\ n\}$ viewed as one whose arguments proceed not by finite differences but by differentials is a case of Gauss' function F $(\alpha,\ \beta,\ \gamma,\ x)$, viz., that case where $\alpha=-m$, $\beta=-n$, $\gamma=x=\infty$.

BISHOPTON, GLASGOW,
7th Nov. 1884.

# On the number of conditions determining geometrical figures.

By A. Y. **Fraser**, M.A.

§ 1. This paper is an attempt (I) to deduce from first principles the number of conditions required to determine a plane polygon of $n$ sides; (II) thence to deduce the numbers for special cases; and (III) to discuss the effects of a redundancy and a deficiency in the number of conditions. An investigation of this kind should form an important as well as interesting accompaniment to the ordinary study of elementary geometry.

§ 2. Figures may agree in many respects, as in perimeter, size of angles,* &c.; but here they are to be considered only as they agree or differ in (*a*) Shape, (*b*) Size, (*c*) Position; or combinations of these.

When figures are to be determined in position, a plane is assumed containing axes of reference.

§ 3. A condition is the limitation to which an element or part of a figure is to be subjected. This limitation is always ultimately reducible to an assertion of equality between an element of the figure and a given magnitude of the same kind. The given magnitude may be (*a*) A ratio, (*b*) an angle, (*c*) a length; or any known function of these.

In enumerating conditions certain precautions must be taken that spurious conditions be not admitted. Conditions must be consistent, otherwise the problem is impossible; but they must in addition stand the two tests now to be discussed. One of these tests will be seen to relate to the subjects, the other to the predicates of conditions.

§ 4. *In the first place in each condition the element given must have a determinate and stated relation to the figure to be constructed.* In other words the number of conditions is in every case deduced on the supposition that the points, lines, angles or parts of figures mentioned are described in language free from ambiguity.

For example, it is not in general sufficient to say "Given two sides of a quadrilateral"; the sides should be further described as either "adjacent" or "opposite" as the case may be. So in general it is not sufficient to say "Given a diagonal"; the particular diagonal should be indicated. Whether a condition satisfies this requirement may always be tested thus: Imagine the figure determined, and see if the condition in question be now determined; if so, and only if so, it is to be admitted as in this respect a proper condition.

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* In crystallography, for example, angular relations are the all-important thing, not linear relations.

*In the second place the conditions must be independent.* It should not be possible to deduce any one condition from the others.

Without actually trying to deduce a suspected condition from the others, the following test is often of easy application: A condition that is independent, and only such a condition, may be varied without in any way changing the others. As an example, apply this test to the triangle having given first the three sides (independent), second the three angles (dependent).

## I.

### § 5. The point.

Of a point the position is all that has to be determined, and this is done by two conditions. This number may be deduced from the conditions for the straight line in much the same way as will be done for an infinite line; or, independently, from the fact that a point is determined by two coördinates parallel to the axes.

It will be observed that the set of conditions for a point being determined, the set for a line can be deduced; and reciprocally. No matter then which of the two we begin with, a good reason can be given for beginning with the other. To those who think the point has an irresistible claim to priority it may be remarked that to determine a point by its coördinates $(a, b)$ is really to determine it by the intersecting of two lines whose equations are $x = a$, $y = b$.

### § 6. The straight line.

*The number of conditions determining the shape of a straight line is zero.* For the shape is fixed by the definition.

*The magnitude of a straight line is determined by one condition.* For the length only is required.

*The position of a line of given magnitude is determined by three conditions.* These may be the intercepts OA, OB, on the axes, and the distance of one end of the line from the origin; but there are many more sets of three equally suitable.

Thus a finite straight [illegible] completely determined by four conditions. Hence an infinite straight line is determined by two conditions; for the characteristic "infinite" involves two conditions, first the distance of one end from the origin (infinite), and second the length of the line (infinite). Compare this with the $y = mx + c$ of coördinate geometry, where the determining conditions are $m$ and $c$.

7. An important principle, to be used later on, has to be stated here. If a figure be already determined in shape, its size will be determined if one line of the figure be given in length, and then its position if one line be given in position. Hence in any figure,

| | | | | |
|---|---|---|---|---|
| if | $N$ | = number of conditions for | | shape, |
| then | $N+1$ | = | ,, ,, | shape + size, |
| and | $(N+1)+3$ | = | ,, ,, | shape + size + position. |

8. The triangle.

*The shape of a triangle is determined by two conditions.*

If there be two triangles ABC and PQR, and if $PQ = k \cdot AB$, $QR = k \cdot BC$ and $RP = k \cdot CA$; then by giving $k$ in succession all possible values we get the series of triangles having what we recognise as similarity of shape to ABC. By eliminating $k$ we get the two independent conditions $PQ : QR = AB : BC$, and $QR : RP = BC : CA$. This and other methods of deducing the number of conditions are of course to be found in the Sixth Book of Euclid. It need hardly be pointed out that among the conditions for similarity of shape a length can never occur.

*The shape and size of a triangle are determined by three conditions.*

*The shape, size, and position of a triangle are determined by six conditions.*

These results might be got from § 7 by putting $N = 2$; but they can be got independently in a great variety of ways. For example, the magnitude and position of the side AB of a triangle are given by four conditions (§ 6), and other two conditions, $\angle B$ and side BC, will determine the triangle completely.*

§ 9. The polygon.

*The shape of an* n*-gon is determined by* 2n − 4 *conditions.*

This result may be got by dividing the $n$-gon into $n-2$ triangles by lines drawn from a vertex, and applying the first proposition in § 8.

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* It may be of interest to mention that [illegible]castle's *Elements of Geometry*, sixth edition (1818), pp. 416-431, contains a list of one hundred and seventy-one cases of triangles determined by sets of three conditions, and this list might now be indefinitely increased.

It may also be got by noticing that the shape is determined if $n-1$ ratios of consecutive sides be given, and also $n-3$ consecutive angles. And $(n-1)+(n-3)=2n-4$.

*The shape and size of an* n-*gon are determined by* 2n – 3 *conditions.*

*The shape, size, and position of an* n-*gon are determined by* 2n *conditions.*

These results may be obtained as follows:

*a.* From § 7 by putting $N=2n-4$.

*b.* By dividing the $n$-gon into $n-2$ triangles and applying the results of § 8.

*c.* By giving all the sides ($n$) and all the angles save three ($n-3$), and so determining the $n$-gon in shape and size.

*d.* By supposing given the $n$ infinite lines ($2n$ conditions), the intersections of which determine the $n$-gon completely.

*e.* By supposing the $n$ vertices given ($2n$ conditions), which also determine the $n$-gon completely.*

The following problem is discussed in order to show a real and a fictitious set of $2n$ conditions: To construct an $n$-gon, having given the $n$ middle points of the sides ($2n$ conditions). It is known that this is a determinate problem only if $n$ is an odd number. The explanation of this is got on applying the second test of § 4, when it is found that $n$ being even, the $n$ points are not all independent. To prove this let us take the eight-sided polygon ABCDEFGH, the mid points of AB, BC, &c., being A′B′C′D′E′F′G′H′, and try to deduce H′ from the others. AC, CE, EG being parallel to and double A′B′, C′D′, E′F′, are known in direction and magnitude. Hence GA and therefore G′H′ are known in direction and magnitude; and the point G′ is known; and therefore H′ is known. And this will evidently hold whatever the number of sides be so long as it is even. Hence in putting down the $n$ points, if the last one happen to be put down in the right place, the number of independent conditions is deficient, and the problem is indeterminate (as we shall see in § 19); and if the last point be put down in the wrong place the conditions are inconsistent, and the problem is impossible (§ 3). If now a similar method be taken with a figure of an odd number of sides, a side, and not the distance between two mid points, is determined in magnitude and direction; and the construction of the figure follows at once.

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*Throughout this paper the assumption is made that if a set of $m$ conditions determine a figure, any other set of $m$ conditions will determine it. The whole investigation goes to support this assumption; but pending the production of a rigid proof, some considerations in its favour may be offered. Every plane figure is determined by its $n$ angular points, and each of these has two degrees of freedom. Hence a plane figure has $2n$ degrees of freedom. And every condition reduces that number by one, and by one only. Thus for any figure the number of determining conditions is constant.

## II.

§ 10. The numbers we have found hold for polygons, of which the only thing stated in their definition is the number of sides. When figures are more particularly characterized, as parallelograms, regular figures, &c., a number of the conditions are involved in the definition of the figure; and by that number has the general expression to be diminished.

§ 11. Denoting the sides of a figure by $a$, $b$, $c$..., and the angles by A, B, C..., let us take a few examples. In the case of triangles, isosceles triangles have involved in their definition the condition $a = b$, and there remain two conditions required to determine shape and size; right-angled triangles have $\angle$ A a right angle, again leaving two conditions; while equilateral triangles have $a = b$, $b = c$, and there is left only one condition. In the case of quadrilaterals, determined as to shape and size by five conditions ($2n - 3$), there are the special cases trapezium ($a \parallel c$), symmetric trapezium ($a \parallel c$, $b = d$), parallelogram ($a \parallel c$, $b \parallel d$), rectangle ($a \parallel c$, $b \parallel d$, A a right angle), rhombus ($a = b$, $b = c$, $c = d$), and square ($a = b$, $b = c$, $c = d$, A a right angle), which, after the numbers of defining conditions are subtracted, are found to be determined respectively by four, three, three, two, two and one conditions. Examples in verification of these results are to be found everywhere in elementary geometry.

*A regular* n*-gon is determined in shape and size by one condition.* For the equality of the sides involves $n - 1$ conditions, and of the angles $n - 3$, that is $2n - 4$ in all. If this number be deducted from the general expression $2n - 3$, the remainder is one as was asserted. Another way of stating this is to say that the shape is determined by the definition, and one condition will determine the size.

§ 12. So far, mention has been made only of rectilineal figures; but a very curious problem is presented in this subject by the circle. Looked upon as the limiting case of the regular polygon, it might be supposed to be determined in size, like a regular polygon, by one condition. And it is sometimes so determined; for to give the radius determines the circle. But we find in other cases two conditions given, as a chord and the angle it subtends at the centre, or a chord and its distance from the centre; and even three conditions, as two chords meeting in the circumference and the angle between them.

At first sight this state of matters seems to imply a breach of continuity between a regular polygon and its limit; but there is no such breach, as will presently be seen.

§ 13. *The apparent anomalies in the circle arise from the failure to comply with the first requirement of* § 4; a failure that can easily be shown to produce the same results even in regular rectilineal polygons.

§ 14. We shall take as an example to work with the regular dodecagon *(fig.* 10*)*; but the reasoning will apply whatever be the number of sides. This polygon is made up of twelve equal isosceles triangles, and if one of these be known the polygon is determined. To attain this end one condition will suffice; since to determine an isosceles triangle two conditions are necessary, and here one, the vertical angle, is known already. To give then anything, as side, base, perpendicular from vertex on opposite side, having a definite and stated relation to one of the triangles such as AOB will determine the polygon.

§ 15. But suppose now the datum* were, "Given the line joining two vertices of the polygon." This line may be AB, AC, AD, AE, AF or AG—all other cases may be looked upon as repetitions of these—and each of these gives a different polygon. This in fact amounts to giving the base of an isosceles triangle AOB or AOC, &c., whose vertical angle is not definitely known. The vertical angle is some one of those included in $p \times \frac{1}{3} rt \angle$ where $p$ may be 1, 2...6. If then a line be given and merely said to be a diagonal, there is always a number of solutions. This number increases with the number of sides of the polygon, and in the case of the circle, where the diagonal has become the chord, it is infinite. The solution becomes definite if the vertices between which the diagonal is drawn be stated, or, what is the same thing, if the angle the diagonal subtends at the centre of the polygon be given. In an $n$-gon this angle is restricted in magnitude to one of the series $[1, 2, 3, \ldots, n-1] 4 rt \angle s \div n$; a series that approximates as $n$ increases to the case of the circle where any angle up to four right angles may be taken.

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* "A datum is any quantity, condition, or other mathematical premiss which is given in a particular problem."—De Morgan, in *The Penny Cyclopædia*, vol. viii, p. 313.

§ 16. To take yet another case, let it be required to construct a dodecagon from the two data, "Given two diagonals drawn from the same vertex." These two lines belong to some such quadrilateral as ALOD *(fig.* 10*)*, which we are required to determine without five, the proper number of conditions. We know AL, AD, and that OL = OA, OA = OD; but the fifth condition is incomplete, for we know only that $\angle$ LOD is one of some half dozen possible angles. This angle has still to be got either directly, or by the naming of the vertices joined, or by the giving of the angle between the diagonals. Thus we may have as data for a regular polygon, two diagonals drawn from a common vertex, and the angle between them. For every regular polygon these three conditions must be taken from a finite number of possible sets of lines with their included angle. But the greater the number of sides of the polygon the greater is the number of these sets, till in the case of the circle the number of sets is infinite. In other words, any two lines may be given inclined at any angle, and a circle can be described through their extremities; or to put it in yet another way, a circle can be described through any three points.

In much the same way, the sides produced of a regular polygon being considered to correspond to the tangents to a circle, the case of the circle determined by its touching three given straight lines can be deduced and explained.

Thus the apparently redundant conditions are needed to specify completely the element of the one condition really given.

§ 17. One more comparison between a polygon and a circle may be worth mentioning. A regular polygon is determined in size and position if the cöordinates of its centre, the length of a line from the centre to a vertex, and the angle this line makes with one of the axes be given—in all, four conditions. The last condition becomes, if we may so speak, of less importance as the sides of the polygon become more numerous, till in the circle it is dispensed with altogether.

## III.

§ 18. *Redundancy of conditions means impossibility of construction.* For a figure being determined by a certain number of conditions, to give more than that number is to impose conditions we are in general unable to carry out.

There is sometimes an apparent redundancy which does not lead to impossibility of construction. When this happens, all the conditions being properly specified, it is a sure indication that the conditions are not all independent. In fact, if the determining number of conditions for the particular case be taken, then the remaining conditions will be found to be deducible from these. Examples of this may be got in Euclid's treatment of similar triangles, where sometimes (VI. 4, 5) it is made to appear as if three conditions were necessary to determine the shape of a triangle.

§ 19. *Deficiency of conditions results in an indeterminate figure.* In other words, an infinite number of figures can be got to satisfy these conditions. It is from this source we derive Locus problems. If a point or a line of a figure be given in position, and a number of conditions be given less, by one usually, than the determining number, an infinite number of figures will be got varying continuously. Then if a particular point be followed up in all these figures, the curve thus got is the locus of the point.

To speak of 'the curve' is hardly correct, for the locus often consists of two or more curves. We may bring the paper to a conclusion by pointing out how this plurality of curves arises.

§ 20. At the outset the number of conditions determining a line (on which the subsequent results were founded) was deduced on the supposition that a line is symmetrical. But the moment a line is made the side of a figure, lateral symmetry ceases; for now a line has a side towards and a side away from the figure—has an inside and an outside. Consequently when a figure is fixed in position by the fixing of a line the figure will in general have two positions according to the side of the line that is placed towards one of the axes. A similar remark may be made about the end symmetry of a line, and a consequent pair of possible positions.

The four cases occur when we construct on the same base four congruent triangles. And now it will be seen why a locus may consist of more than one curve. For fix the base and the area, and the locus of the vertex is a pair of straight lines. Fix the base and a base angle, and the locus of the vertex is four straight lines. Other examples will suggest themselves.

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*Third Meeting, January 9th,* 1885.

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A. J. G. Barclay, Esq., M.A., President, in the Chair.

## On the problem to construct the minimum circle enclosing $n$ given points in a plane.

### By Professor Chrystal.

If we consider the circle circumscribing any triangle ABC (see figures 11, 12), and diminish its radius still causing it to pass through A and B; then if ACB be an acute angle, C passes without the circle, but if ACB be an obtuse angle, C remains within the circle. If C be a right angle, the radius of the circle, being $\frac{1}{2}$AB, cannot be farther diminished.

This result, which will be used in what follows, leads in the first place to the result, otherwise obvious, that the minimum circle enclosing three points is the circle through them when they form an acute-angled or right-angled triangle, but the circle having as diameter the line joining the two which are most distant when the triangle which they form is obtuse-angled.

Let there be $n$ points. We can always select $m$ of them which form the vertices of a convex polygon enclosing all the others. The problem therefore reduces to finding the minimum circle enclosing a convex $m$-gon.

Since the $m$-gon lies wholly on one side of the unlimited straight line, of which any one of its sides is a part, we may regard this line as a circle of infinite radius wholly enclosing the $m$-gon. Let us diminish the radius of this circle continuously (always supposing it to pass through two particular vertices of the $m$-gon). Then one or other of two things will happen. Either the radius will diminish down to its minimum value (half the selected side of the $m$-gon) before the circle passes over any vertex of the $m$-gon; in which case the circle on the selected side as diameter contains all the $n$ points, and must be the minimum circle, since no less circle can contain the two vertices started with: or the circle will first pass over a third vertex of the $m$-gon. In the latter case if the three vertices involved form an acute-angled triangle, the minimum circle is reached, for this circle contains all the $n$ points and is the least that can contain the three in question; if the triangle be obtuse-angled, we observe first that the obtuse angle cannot be opposite the selected side, for by supposition we have not reached the minimum radius which corresponds to a right angle, the obtuse angle must therefore be at one or other of the extremities of the selected side. Throwing out the vertex at which the obtuse angle occurs, let us still farther diminish the radius, the circle still passing through the two vertices retained; by this process the neglected vertex passes inside the circle and need not be farther considered; and we shall arrive either as before at a minimum circle on the line joining the two retained vertices as diameter, or at a circle passing through three vertices of the polygon, which will be a minimum circle if these vertices form an acute-angled triangle, and may be farther diminished if this triangle be obtuse-angled, it being remarked as before that the obtuse angle cannot be at the vertex last taken in, and so on.

Since the radius of the circle is being continually diminished this process must come to an end, and that in one or other of two ways, viz:—either the minimum circle must appear as a circle on a side or diagonal of the $m$-gon as diameter (which side or diagonal will be the greatest distance between any pair of the $m$ points), or it will appear as a circle described through three of the vertices of the $m$-gon, forming an acute-angled triangle. The minimum circle may in certain limiting cases contain more of the vertices than the two or the three that determine it, but this does not affect the conclusion.

It is interesting topologically to remark that point systems divide themselves into two distinct classes, each of course with limiting cases, according as their minimum circle is determined by two of the points, or by three of them.

The above discussion suggests at once the following practical method for determining the minimum circle of a point system.

Construct by taking $m$ of the points a convex polygon enclosing them all. Take any side of this $m$-gon, and find the vertex at which it subtends the least angle. If this least angle be right or obtuse, the minimum circle is the circle on the chosen side as diameter. If the triangle formed by the three vertices be acute-angled its circumscribed circle is the minimum circle; if not, take the side opposite the obtuse angle, find the vertex at which it subtends the least angle, and go on as before.

The number of steps in the construction will depend of course on the side first selected. It is clear however that no side or diagonal of the polygon can occur more than once as the chosen side, so that $\frac{1}{2}m(m-1)$ is an upper limit to the number of steps.

In any particular case it might be more expeditious first to try whether a circle on the greatest side or diagonal encloses all the points; if this is not so, then the minimum circle is obtained by finding that triangle formed by three of the $m$ points which is acute-angled and has the greatest circumscribing circle.

The theory given for the case of $n$ points in a plane may be extended to $n$ points in space. Consider any three points A,B,C, and a fourth D. Through the four points we may describe a sphere. Let us suppose the sphere always to pass through ABC, and its radius to vary; the circle circumscribing ABC thus forms the base of a variable spherical segment. Through the axis of this circle we can always draw a plane containing D. This plane will cut the sphere in a great circle, and the circle ABC in two points K and L. The segment KDL will be greater than, equal to, or less than, a semicircle, according as the spherical segment is greater than, equal to, or less than, a hemisphere. In the first case, if we diminish the radius of the sphere, D will pass without the spherical segment. Hence if all the four spherical segments are greater than hemispheres, that is if the centre of the sphere is within the tetrahedron ABCD, the sphere passing through ABCD is the minimum sphere. In the second case the radius cannot be further diminished, unless one of the angles (C say) of the triangle ABC be obtuse, and then the sphere may be reduced to a sphere having the circle described about ABD as a great circle. If the triangle ABD is acute-angled, the sphere now got is the least. If ABD is obtuse angled, the sphere may be finally reduced to the sphere having as diameter the side opposite the obtuse angle. In the last case, if the radius of the sphere be diminished, D will pass within; and the possibility of further reductions may be discussed as in the second case.

We may therefore classify point systems in space of three dimensions into three classes according as the minimum sphere is determined by four, three, or two of the points. A chain of construction for finding the minimum sphere may be given analogous to that for the minimum circle in the case of co-planar points; but the details are naturally more complicated.

An analytical theory of the above geometrical relations doubtless exists, and it might be extended to space of $n$ dimensions.

The plane problem might be generalised in various ways. For example, if we project orthogonally the $n$ co-planar points upon another plane, we have the problem: To find the ellipse of minimum area enclosing $n$ given points, which shall be similar and similarly situated to a given ellipse.

The problem of the circle, or sphere, enclosing $n$ points is interesting in connection with the theory of functions of a complex variable, and in the allied subject of the Theory of the potential. There are also certain questions of a practical nature regarding the representation of observations by curves of a given description which lead to similar problems. Let us suppose for example that the points in figure 13 represent the plotted results of a series of observations, and that we have some reason to suppose that the points would lie on an ellipse whose foci are F and F′, if the results were freed from errors of observation. If now we draw the least ellipse having F and F′ as foci which includes all the points, and the greatest confocal ellipse which excludes them all, we can assign the utmost error that can be incurred by representing the observations by means of an ellipse having F and F′ as foci; and can thus tell whether, in view of the possible errors of observation, such an ellipse is a sufficiently accurate representation or not.

It was a practical question of the above nature that first suggested to me the problem of the minimum circle enclosing $n$ points. I drew Professor Tait's attention to it, and he gave an interesting topological discussion of the case of four points, in an address delivered before this society, and published in the *Philosophical Magazine* for January 1884 (Fifth Series, vol. xvii., pp. 30-46). In consequence of the interest shown in the problem by Professor Hermite, I worked out for myself and sent to him the solution above given. I learned a day or two before communicating it to the Mathematical Society of Edinburgh that the problem had originally been proposed by Professor Sylvester (*Quarterly Journal of Mathematics*, I., p. 79); and that a solution had been given by him in an article in the *Philosophical Magazine* more than twenty years ago (1860, Fourth Series, vol. xx., pp. 206-212). I have since consulted this paper, and find that the solution there given is due to Peirce. It is only briefly indicated, but appears to be substantially identical with the one I have given above. An extension is given to points on the surface of a sphere; but no mention is made of the three dimensional problem. The connection between the minimum circle problem and the main part of Sylvester's article, which deals with the approximate representation of $\sqrt{x^2+y^2}$ and $\sqrt{x^2+y^2+z^2}$ by means of linear functions of $x$, $y$, $z$, is very interesting.

# Note on the equation connecting the mutual distances of four points in a plane.

By Thomas Muir, LL.D.

1. If the distances (12), (13), (14), (23), (24), (34) between four points 1, 2, 3, 4 on the circumference of a circle be denoted by $a$, $b$, $c$, $d$, $e$, $f$ respectively, then a certain relation (A) is known to connect $a$, $b$, $c$, $d$, $e$, $f$. The same four points, however, being points in a plane, there subsists between their mutual distances another relation (B). Now, it occurs to one that from these two relations some deduction ought to be possible regarding the mutual distances of four points on a circumference, and the problem is suggested of making the said deduction.

2. Let us first consider the relations (A) and (B) in the forms so elegantly arrived at by Cayley,[*] viz. :—

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 \\ a^2 & 0 & d^2 & e^2 \\ b^2 & d^2 & 0 & f^2 \\ c^2 & e^2 & f^2 & 0 \end{vmatrix} = 0, \qquad \text{(A)}$$

and

$$\begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a^2 & b^2 & c^2 \\ 1 & a^2 & 0 & d^2 & e^2 \\ 1 & b^2 & d^2 & 0 & f^2 \\ 1 & c^2 & e^2 & f^2 & 0 \end{vmatrix} = 0 ; \qquad \text{(B)}$$

and ask what conclusion can be deduced from them. Observing that the first determinant is the complementary minor of the element in the place (1, 1) of the second determinant, we recall a theorem which gives us a result at once. The theorem[*] is—*An axisymmetric deter-*

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[*] *Cambridge Mathematical Journal*, II., pp. 267-271.

[*] This theorem seems to have been first enunciated by Salmon in the 1st edition of his *Lessons Introductory to the Modern Higher Algebra*, p. 124 (Dublin, 1859). Hesse gave it in Crelle's Journal, LXIX., p. 321 (1868), with the foot-note—"Den Satz (8) findet man von Herrn Weierstrass bewiesen in dem "Monatsberichte der Königl. Akademie der Wissenschaften zu Berlin, 4 "März 1858, p. 211. Denn seine von einem Factor abgelöste Function "$\delta_\mu = \Sigma_{\alpha\beta} f(s_\mu)_{\alpha\beta} \Phi_\alpha \Phi_\beta$ welche er als das Quadrat einer linearen function "darstellt, ist, wenn man $a_o^n = \Phi_o$, $a_1^n = \Phi_1$, ...., $a_{n-1}^n = \Phi_{n-4}$ und $a_n^n = 0$ "setzt, gerade die symmetrische Determinante A deren Unterdeterminante "$B = f(s_\mu)$ verschwindet."

*minant whose first element has a vanishing complementary minor is, when changed in sign, expressible as an exact second power.* By reason of the existence of (A) it follows then that (B) can be put in the form

$$-\left\{\begin{vmatrix}0 & d^2 & e^2\\ d^2 & 0 & f^2\\ e^2 & f^2 & 0\end{vmatrix}^{\frac{1}{2}}+\begin{vmatrix}0 & b^2 & c^2\\ b^2 & 0 & f^2\\ c^2 & f^2 & 0\end{vmatrix}^{\frac{1}{2}}+\begin{vmatrix}0 & a^2 & c^2\\ a^2 & 0 & e^2\\ c^2 & e^2 & 0\end{vmatrix}^{\frac{1}{2}}+\begin{vmatrix}0 & a^2 & b^2\\ a^2 & 0 & d^2\\ b^2 & d^2 & 0\end{vmatrix}^{\frac{1}{2}}\right\}^2=0,$$

whence we have

$$\sqrt{d^2e^2f^2}+\sqrt{b^2c^2f^2}+\sqrt{a^2c^2e^2}+\sqrt{a^2b^2d^2}=0 \qquad \text{(I)}$$

as the result desired. The ambiguities of sign which occur in it, and which are a little troublesome to disentangle, detract somewhat from the value of this mode of proceeding.

3. The old form of (A) is

$$(af+be+cd)\,(-af+be+cd)\,(af-be+cd)\,(af+be-cd)=0\,; \qquad (\mathrm{A}_2)$$

and this, it is manifest, viewed apart from its connection with (B), is much more instructive than the form $(\mathrm{A}_1)$. No corresponding non-determinant form of $\mathrm{B}_1$ having like advantages, has, so far as I am aware, been proposed. The following is therefore suggested—

$$\begin{aligned}d^2e^2f^2+b^2c^2f^2+a^2c^2e^2+a^2b^2d^2&-(a^2+f^2)\,(-a^2f^2+b^2e^2+c^2d^2)\\ &-(b^2+e^2)\,(a^2f^2-b^2e^2+c^2d^2)\\ &-(c^2+d^2)\,(a^2f^2+b^2e^2-c^2d^2)=0. \quad (\mathrm{B}_2).\end{aligned}$$

By one who remembers $(\mathrm{A}_2)$ this also can be easily remembered. Its identity with $(\mathrm{B}_1)$ is established by expanding the determinant in $(\mathrm{B}_1)$ according to binary products of the elements of the first row and first column, and uniting those of the resulting determinants which have a common factor.

Now, taking these forms $(\mathrm{A}_2)$ $(\mathrm{B}_2)$, let us see with what readiness an unambiguous result is deducible.

When $(\mathrm{A}_2)$ holds,

$$\begin{aligned}\text{either}\quad & af+be+cd=0,\\ \text{or}\quad & -af+be+cd=0, \quad \text{(Ptolemy's theorem)}\\ \text{or}\quad & af-be+cd=0,\\ \text{or}\quad & af+be-cd=0.\end{aligned}$$

If it be that

$$af+be+cd=0,$$

$$\left.\begin{array}{ll}\text{then} & -a^2f^2+b^2e^2+c^2d^2=-2becd \\ \text{and} & a^2f^2-b^2e^2+c^2d^2=-2afcd \\ \text{and} & a^2f^2+b^2e^2-c^2d^2=-2afbe\end{array}\right\}$$

and manifestly we may extract the square root of both sides of $(B_2)$ and obtain the result

$$def+bcf+ace+abd=0.$$

If, secondly, $-af+be+cd=0,$

then $def+bcf-ace-abd=0$ or $\frac{a}{f}=\frac{bc+de}{bd+ce}.$

If, thirdly, $af-be+cd=0,$

then $def-bcf+ace-abd=0$ or $\frac{b}{e}=\frac{ac+df}{ad+cf}.$

If, fourthly, $af+be-cd=0,$

then $def-bcf-ace+abd=0$ or $\frac{c}{d}=\frac{ab+ef}{ae+bf}.$

The "proportion" form of these identities is the familiar one which results from geometrical demonstration.

4. It is interesting to note that that when $e$ and $f$ are both put equal to $d$, the sides of the equation (B) become divisible by $d^2$, and the equation takes the form

$$\Sigma a^4=\Sigma a^2b^2. \qquad \text{(II.)}$$

What is the import of this algebraic symmetry, when there is no symmetry apparent in the geometry? The answer to this is best seen by looking upon $\Sigma a^4=\Sigma a^2b^2$ as the equation to which we are led in trying to solve the geometrical problem—

*Given the distances* a, b, c *of a point from the vertices of an equilateral triangle: find the side* d *of the said triangle.*

The symmetry then shows that we should be led to the same equation if the problem were—

*Given an equilateral triangle whose side is* a *(or* b, *or* c*) and a point whose distances from two of the vertices of the triangle are* b, c *(or* c, a*; or* a, b*): find the distance of the point from the third vertex.* Hence if we can solve the one problem we can solve the other. But the latter viewed as a geometrical problem requires *no* solution. Consequently the solution of the former is merely the construction indicated in the data of the latter. That is to say, the construction is—*Form a triangle whose sides are* a, b, c*: describe an equilateral triangle on* *any one of them; and join the vertices which are not common to the two triangles.*

The correctness of the construction is at once established by describing an equilateral triangle on the line found and using **Euc.** I., 4.

**5.** As the equilateral triangle of the construction has two possible positions the problem has two solutions, the point from which the distances are measured being in the one case inside the triangle found, and in the other outside. Let us see what light the algebraical equation $\Sigma a^4 = \Sigma a^2 b^2$ throws on this dualism.

Arranging the equation as a quadratic is $d^2$ we have

$$d^4 - (a^2+b^2+c^2)d^2 + (a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2) = 0,$$

whence we find

$$2d^2 = a^2+b^2+c^2 \pm \sqrt{3(2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4)}.$$

But $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$ is $16\Delta^2_{abc}$, if we denote by $\Delta_{abc}$ the area of the triangle whose sides are $a$, $b$, $c$. Hence

$$2d^2 = a^2+b^2+c^2 \pm 4\Delta_{abc}\sqrt{3},$$

and therefore

$$\tfrac{1}{2}d^2\sqrt{3} = \tfrac{1}{4}a^2\sqrt{3} + \tfrac{1}{4}b^2\sqrt{3} + \tfrac{1}{4}c^2\sqrt{3} \pm 3\Delta_{abc}.$$

Using now $d_1$ and $d_2$ to stand for the two values of $d$, and writing $\Delta_a$ for the area of the equilateral triangle whose side is $a$, *i.e.*, for $\frac{1}{4}a^2\sqrt{3}$ we have

$$2\Delta_{d_1} = \Delta_a + \Delta_b + \Delta_c + 3\Delta_{abc},$$

and

$$2\Delta_{d_2} = \Delta_a + \Delta_b + \Delta_c - 3\Delta_{abc};$$

and hence, by addition and subtraction, the theorems

$$\Delta_{d_1} + \Delta_{d_2} = \Delta_a + \Delta_b + \Delta_c, \qquad \text{(III).}$$

$$\Delta_{d_1} - \Delta_{d_2} = 3\Delta_{abc}; \qquad \text{(IV).}$$

that is to say, *The sum of the two equilateral triangles, each of which has its vertices at three given distances from a fixed point is equal to the sum of the equilateral triangles described on the distances; and the difference of the said pair of equilateral triangles is equal to three times the triangle whose sides are the given distances.*

**6.** When in (B) we put $e = d$ and $f = \rho d$ (which of course is a more general case than that of §§ 4, 5) the equation is also symmetrical, but now only with respect to two of the letters involved. Consequently with proper restrictive changes we can solve a more general geometrical problem than that of § 4, viz., where the triangle instead of being equilateral has two sides equal and enclosing a definite angle. We must now know which of the three given distances is the one drawn to the meeting point of the equal sides of the triangle; and taking this distance we make it one of the equal sides of an isosceles triangle similar to that specified, and on the third side describe a triangle with its two other sides equal to the remaining two given distances, and as before join the vertices not common to the two triangles.*

Bishopton, Glasgow,
26th Dec. 1884.

## Geometrical Notes.

### By J. S. Mackay, M.A.

I. A straight line KK′ meets the circumference of a circle at two real or two imaginary points K, K′, and H is the middle point of the real or imaginary chord KK′. If A, B, C, D be any four points on the circumference, and the pairs of straight lines AB,DC, AC,BD, AD,CB meet KK′ at the pairs of points E,E′, F,F′, G,G′; then if any one pair of points be equidistant from H, the two other pairs will also be equidistant.

To prove that E,E′ are equidistant from H, if F,F′ are.

*First Demonstration.* (*Figures* 14, 15.)

Through A draw AA′ parallel to KK′; join A′E′, A′F′, and since E′, F′ are on DC, DB, join A′D.

Then AFF′A′ may be proved to be a convex or crossed isosceles and therefore cyclic trapezium, having A′F′ = AF, and angle A′F′E′ = angle AFE.

---

* Professor Chrystal pointed out that a particular case of this, viz., where the triangle is isosceles *right-angled* is dealt with in the *Annals of Mathematics*, I., p. 24, and Mr Fraser has since received from Dr Rennet, of Aberdeen, a reference to Thomas Simpson's Algebra, 2nd edition, (1755) p. 369, where a very general problem of this nature is stated and solved.

Now angle $A'AC$ is supplementary to angle $A'DC$,
and angle $A'AC$ is equal to angle $A'F'E'$;
therefore angle $A'F'E'$ is supplementary to angle $A'DC$;
therefore the points $A'$, $F'$, $E'$, $D$ are concyclic;
therefore angle $F'A'E' =$ angle $F'DE'$,
$=$ angle $FAE$.
Hence triangles $A'E'F'$ and $AEF$ are congruent, and $E'F' = EF$, and consequently $HE' = HE$.

The proof that $G$, $G'$ are equidistant from $H$, if $F$, $F'$ are, is similar to the preceding.

### *Second Demonstration.* (*Figures* 14, 15.)

Let $AC$, $BD$, on which are $F$, $F'$, meet at $L$.

Consider the triangle $LFF'$ as cut by the transversals $AB$, $CD$, which determine the points $E$, $E'$.

We obtain $\quad LA \cdot FE \cdot F'B = FA \cdot F'E \cdot LB$,
and $\quad LC \cdot FE' \cdot F'D = FC \cdot F'E' \cdot LD$;
therefore, by multiplication and noting that $LA \cdot LC = LB \cdot LD$,
we have $FE \cdot FE' \cdot F'B \cdot F'D = FA \cdot FC \cdot F'E \cdot F'E'$.
Now $\quad F'B \cdot F'D = F'K \cdot F'K' = FK' \cdot FK = FA \cdot FC$;
therefore $\quad FE \cdot FE' = F'E \cdot F'E'$;
therefore $\quad FE : F'E = F'E' : FE'$;
therefore $\quad FE : FF' = F'E' : FF'$.
Hence $FE = F'E'$, and consequently $HE = HE'$.

It may be proved that $FE \cdot FE' = F'E \cdot F'E'$, without introducing the segments $FK, FK'$, $F'K, F'K'$, which in figure 15 are imaginary, in the following way:—

Because $\quad FE \cdot FE' \cdot F'B \cdot F'D = FA \cdot FC \cdot F'E \cdot F'E'$,
therefore $\quad FE \cdot FE' : F'E \cdot F'E' = FA \cdot FC : F'B \cdot F'D$,
$= \pm(r^2 - OF^2) : \pm(r^2 - OF'^2)$,
where $r$ denotes the radius of the circle, and the upper sign is to be taken in figure 14, the lower sign in figure 15.
Now $OF = OF'$, since $F, F'$ are equidistant from $H$;
therefore $\quad FE \cdot FE' = F'E \cdot F'E'$.

### *Third Demonstration.* (*Figure* 14.)

Join $BK$, $BK'$, $CK$, $CK'$.

Since the pairs of angles subtended at $B$ and $C$ by the arcs $KA$, $AD$, $DK'$ are equal;

therefore the pencils B·KADK′ and C·KADK′ are superposable; therefore the anharmonic ratio of B·KADK′ is equal to the anharmonic ratio of C·KADK′,
that is, KK′ · EF′ : KE · F′K′ = KK′ · FE′ : KF · E′K′.
Now since F′K′ = KF, this proportion becomes

EF′ : KE = FE′ : E′K′;

therefore EF′ : F′K = FE′ : FK′.
But F′K = FK′; therefore EF′ = FE′.
Hence E and E′ are equidistant from F and F′, and consequently from H.

The theorem to which Mr James Taylor called attention last session (see *Proceedings of the Edinburgh Mathematical Society* for 1883-4, p. 4) is easily seen to be a particular case of the foregoing.

II. Between two sides of a triangle to inflect a straight line which shall have given ratios to the segments of the sides between it and the base.

Let ABC be the triangle, and let $p : q$ and $r : q$ be the ratios of the segments of the sides to the inflected straight line.

*First Method.* (*Figure* 16.)

From BA cut off BD $= p$; through D draw DE parallel to BC. Cut off CF′ $= r$; with centre F′ and radius $= q$, cut DE or DE produced at the points G′; and join F′G′. Let CG′ meet AB or AB produced at G, and draw GF parallel to G′F′. GF is the line required.

For through G′ draw G′B′ parallel to GB.

Then B′G′ = BD $= p$.

Now, since the quadrilaterals CB′G′F′, CBGF are similar, and either similarly or oppositely situated, C being their centre of similitude; and since B′G′ : G′F′ : F′C $= p : q : r$;
therefore BG : GF : FC $= p : q : r$.

*Second Method.* (*Figure* 17.)

Take BD such that AC : BD $= r : p$; and through D draw DE parallel to BC. With centre A and radius AG′, such that AC : AG′ $= r : q$, cut DE or DE produced at the points G′; and join AG′. Let CG′ meet AB or AB produced at G, and draw GF parallel to G′A. GF is the line required.

For through G′ draw G′B′ parallel to GB.

Then B′G′ = BD; therefore AC : B′G′ = $r : p$.

Now since the quadrilaterals CB′G′A, CBGF are similar, and either similarly or oppositely situated, C being their centre of similitude; and since B′G′ : G′A : AC = $p : q : r$;
therefore BG : GF : FC = $p : q : r$.

*Third Method.* (*Figure* 18.)

Take AD′ such that AC : AD′ = $r : p$. With centre D′ and radius D′B′, such that AC : D′B′ = $r : q$, cut BC or BC produced at the points B′; and join D′B′, D′C. Through B draw BD parallel to B′D′ to meet CD′ or CD′ produced at D; through D draw DF parallel to AB to meet AC or AC produced at F; and through F draw FG parallel to BD to meet AB or AB produced at G. GF is the line required.

For BG = DF, and GF = BD.

Now since the quadrilaterals CB′D′A, CBDF are similar and either similarly or oppositely situated, C being their centre of similitude; and since AD′ : D′B : AC = $p : q : r$;
therefore FD : DB : FC = $p : q : r$;
therefore BG : GF : FC = $p : q : r$;

With these three methods, which are essentially the same, it may be interesting to compare *Proceedings of the Edinburgh Mathematical Society* for 1883-4, p. 27; *Educational Times*, vol. 37, p. 328; Vuibert's *Journal de Mathématiques Elémentaires*, 9$^e$ année, p. 45.

The following solution of the problem, in the case when $p$, $q$, $r$ are equal, is due to Mr Robert John Dallas.

(*Figure* 19.)

Suppose the thing done.

Then since BD = ED, $\angle$ ADE = 2 $\angle$ DBE.
Similarly $\angle$ AED = 2 $\angle$ ECD.

$$\begin{aligned} \text{Now} \quad & \angle ADE + \angle AED = \angle ABC + \angle ACB; \\ \text{therefore} \quad & \angle DBE + \angle ECD = \tfrac{1}{2}(\angle ABC + \angle ACB); \\ \text{therefore} \quad & \angle OBC + \angle OCB = \tfrac{1}{2}(\angle ABC + \angle ACB), \\ & = \text{a constant angle}; \end{aligned}$$

therefore O lies on the circumference of a known circle.
If O could be found on this circumference, so that BD might be equal to CE, the problem would be solved; for in proving O to lie on this circumference it was assumed that BD = DE.

Draw CF parallel to AB. Take any point G in AC, make CH = CG, and join GH cutting BC in K.
On GH describe a segment of a circle containing an angle equal to $\frac{1}{2}(\angle ABC + \angle ACB)$, and let the arc of the segment cut BC at R. Through B draw BE parallel to RG, and BF parallel to RH.

Then by similar triangles it will follow that
$EC = FC$, and $\angle EBF = \angle GRH = \frac{1}{2}(\angle ABC + \angle ACB)$.
Let BE meet at O the arc of the segment described on BC, and containing an angle equal to the supplement of $\frac{1}{2}(\angle ABC + \angle ACB)$.
Then $\angle EOC = \frac{1}{2}(\angle ABC + \angle ACB) = \angle EBF$;
therefore BDCF is a parallelogram, and $BD = FC$.
Now EC has been proved equal to FC, and O is on the circumference of the known circle;
therefore $BD = DE = EC$.

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*Fourth Meeting, February 13th,* 1885.

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A. J. G. Barclay, Esq., M.A., President, in the Chair.

## Note on a Plane Strain.

### By Professor Tait.

The object of this note is to point out, by a few remarks on a single case, how well worth the attention of younger mathematicians is the *full* study of certain problems, suggested by physics, but limited (so far as that science is concerned) by properties of matter.

In de St Venant's beautiful investigations of the flexure of prisms, there occurs a plane strain involving the displacements

$$\xi = \frac{xy}{D}, \quad \eta = \frac{y^2 - x^2}{2D}.$$

Physically, this is applicable to de St Venant's problem only when $x$ and $y$ are each small compared with D. But it is interesting to consider the results of extending it to all values of the cöordinates. This I shall do, but very briefly.

1. The altered cöordinates of any point are given, in terms of the original cöordinates, by

$$x'=x\left(1+\frac{y}{\mathrm{D}}\right),\ y'=y+\frac{y^2-x^2}{2\mathrm{D}}.$$

Hence

$$\delta x'=\delta x\left(1+\frac{y}{\mathrm{D}}\right)+\delta y\frac{x}{\mathrm{D}},$$

$$\delta y'=-\delta x\frac{x}{\mathrm{D}}+\delta y\left(1+\frac{y}{\mathrm{D}}\right).$$

From these we see at once that, so far as an indefinitely small area is concerned, the strain is a mere extension in all directions in the ratio

$$\sqrt{\left(1+\frac{y}{\mathrm{D}}\right)^2+\frac{x^2}{\mathrm{D}^2}}:1,$$

combined with a rotation through an angle whose tangent is

$$-\frac{x}{\mathrm{D}+y}.$$

2. Hence elementary squares remain squares; and any two series of lines, dividing the plane into little squares, will continue to do so after the strain.

One simple case is furnished by sets of lines parallel to the axes. Thus $y=b$ becomes the parabola

$$x'^2=-\frac{2\mathrm{D}^3}{(\mathrm{D}+b)^2}\left(y'-b-\frac{b^2}{2\mathrm{D}}\right),\quad \dots\quad \dots\quad (1.)$$

and $x=a$ becomes a parabola

$$x'^2=\frac{2a^2}{\mathrm{D}}\left(y'+\frac{a^2+\mathrm{D}^2}{2\mathrm{D}}\right).\quad \dots\quad \dots\quad (2.)$$

These groups of parabolas, (1) and (2), must evidently be othogonal, and if the simultaneous small increments of $a$ and $b$ be *equal*, must divide the plane into little squares. But, as it is clear from (2) that the sign of $a$ is immaterial, the two lines

$$x=a,\ x=-a$$

are *both* deformed into the same parabola. Hence it appears that every part of the area becomes *duplex*. This will be examined by another and more suitable method later.

Having thus obtained another set of lines which divide the plane into squares, we may begin again with it and obtain a third set, &c.

3. A line, $y = mx$, passing through the origin, becomes the parabola

$$\left(\frac{m}{m^2-1}y' - \frac{x'}{2}\right)^2 = \frac{2D}{m^2+1}(mx' - y').$$

The orthogonal trajectories of all such parabolas are the curves into which the circles

$$x^2 + y^2 = c^2$$

are deformed. Their equation may be put in the form

$$\sqrt{x'^2 + y''^2} - \frac{c}{2} = \pm \sqrt{\frac{c^2 y''}{D} + \frac{c^2}{4}},$$

where $y''$ is written instead of $y' + \frac{c^2}{2D}$.

These curves have the property that, at every point, *the sum (or difference) of the distance from a given point, and of a multiple of the square root of the distance from a given line, is constant.*

4. But, if we express the new rectangular cöordinates of a point in terms of its original polar cöordinates, we have

$$x' = r\cos\theta + \frac{r^2}{2D}\cos\left(2\theta - \frac{\pi}{2}\right),$$

$$y' = r\sin\theta + \frac{r^2}{2D}\sin\left(2\theta - \frac{\pi}{2}\right).$$

Thus the deformed circles, above spoken of, are seen to be *epicycloids of the cardioid series.* Their orthogonal trajectories are the parabolas just mentioned.

5. Another curious set of questions is, as it were, the reverse of these:—*i.e.*, what were the curves, in the unstrained plate, which became the system

$$x = a, \; y = b,$$

or the other (also orthogonal) system

$$y = mx, \; x^2 + y^2 = c^2 ?$$

6. But a different transformation is still more explicit in the information it gives. Shift the origin to $(0, -D)$, and we have

$$x' = \frac{xy}{D}, \; y' = \frac{y^2 - x^2 + D^2}{2D}.$$

If we put $x = \rho\sin\phi$, $y = \rho\cos\phi$, these give

$$x' = \frac{\rho^2}{2D}\sin 2\phi, \; y' - \frac{D}{2} = \frac{\rho^2}{2D}\cos 2\phi.$$

Hence a circle, of radius $\rho$, surrounding the new origin, becomes a circle of radius $\frac{\rho^2}{2\mathrm{D}}$ surrounding the point $\left(0,\ -\frac{\mathrm{D}}{2}\right)$ half-way between the new and old origins. The $\phi$ of any point in the circle becomes $2\phi$.

Hence the whole surface is opened up like a fan round the new origin, *every* radius through this origin having its inclination to the axis of *y doubled.* Thus the parts of a diameter, on opposite sides of the centre, are brought to coincide; and an infinitely extended line, through the centre, becomes limited at the centre. Thus what was a single sheet becomes duplex, as was said above.

7. It suffices to have indicated, by a partial examination of some of the curious features of a single case, the stores of novelties which are thus easily reached. See especially, for additional materials of the same kind, the investigation in §§ 706-7 of Thomson and Tait's *Natural Philosophy.*

38 George Square, Edinburgh,
*15th January* 1885.

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## Boole's and other proofs of Fourier's Double-Integral Theorem.

By Peter Alexander, M.A.

In my former paper on Fourier's double-integral I remarked that Poisson's form of the integral gave the same incorrect result as Fourier's form in an example by which I tested it, and seemed subject to the same limitations.

This, I now find, is not the case. The assertion, though quite true of the form

$$f(x) = \mathop{\mathrm{L}}_{\kappa=0} \frac{1}{\pi}\int_0^{\infty} dq \int_{-\infty}^{\infty} d\alpha\, \epsilon^{-\kappa q} \cos(qx - q\alpha) . f(\alpha)$$

which I then dealt with, and which I understood to be Poisson's formula, having it on the authority of Freeman (*Fourier's Theory of Heat*, p. 351), is not true of the correct form.

Dr Muir having a difficulty in accepting my statement, asked me to reconsider the matter, and kindly referred me to several papers on

Fourier's theorem. After perusing these I have come to the conclusion that Poisson's formula is

$$f(x) = \mathbf{L}_{\kappa=0}\frac{1}{\pi}\int_{-\infty}^{\infty} d\alpha f(\alpha)\int_{0}^{\infty} dq\, \epsilon^{-\kappa q}\cos(qx - q\alpha),$$

and that my remarks do not apply to it. I have in fact tested it by the example I used as a test in my former paper, and find that it stands the test, provided $\kappa$ be not put equal to 0 immediately after the integration with respect to $q$, but after the integration with respect to $\alpha$.

It having been suggested that I might subject to a critical examination the proofs in the papers to which I had been referred, I now proceed to do so.

The first, and in my opinion the most important of these papers, is one by Boole in the *Transactions of the Royal Irish Academy*, vol. xxi., pp. 124-130, entitled "On the Analysis of Discontinuous Functions." The other papers are those by J. W. L. Glaisher and by G(regory ?) referred to in a foot-note to my former paper.

Boole's proof is beautiful and masterly, and every one interested in this subject should read it. But though the result he arrives at is correct, I have found a serious error at a certain point of it. To render intelligible what I have to say, a sketch of the proof must be given. It runs thus :—

"If in the function $\tan^{-1}\frac{a-x}{\kappa}$ we suppose $x < a$ and $\kappa$ a positive "quantity, then as $\kappa$ is diminished the limit of the values of the func- " tion will be $\frac{\pi}{2}$. This is evident.

" If $x = a$, the limit is 0, the entire series of values being 0.

" If $x > a$, the limit is $-\frac{\pi}{2}$.

" Let $\Delta f(a) = f(a + \Delta a) - f(a)$. Then

$$\Delta\tan^{-1}\frac{a-x}{\kappa} = \tan^{-1}\frac{a+\Delta a-x}{\kappa} - \tan^{-1}\frac{a-x}{\kappa}, \qquad \text{(I)}$$

and applying what precedes to each term of the second member we find that the limit of their sum is $\pi$, or $\frac{\pi}{2}$, or 0, according as $x$ lies be-

" tween $a$ and $a+\Delta a$, or is equal to $a$ or $a+\Delta a$, or lies entirely without " these limits.

" In what follows we shall suppose that $\kappa$ is thus diminished, so " that by any expression involving $\kappa$ we shall understand the limit to " which it approaches as $\kappa$ approaches to 0. Then

$$\frac{1}{\pi}\left(\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa}\right)f(x)=f(x),\ \text{or}\ \frac{f(x)}{2},\ \text{or}\ 0,\qquad\text{(II)}$$

" according as $x$ lies between, upon, or without the limits $a$ and $a+\Delta a$."

Thus far the reasoning is unexceptionable, but what now follows is faulty. It runs thus:—

" When $\Delta a$ becomes infinitesimal it may be replaced by $da$, and " the symbol $\Delta$ by $d$, whence by (I) and its consequences,

$$d\tan^{-1}\frac{a-x}{\kappa}=\pi,\ \text{or}\ \frac{\pi}{2},\ \text{or}\ 0,\qquad\text{(III)}$$

" according as $x$ lies between, upon, or without the limits $a$ and $a+da$.

" Effecting the operation in the first member, we have, under the " same conditions,

$$\frac{\kappa da}{\kappa^2+(a-x)^2}=\pi,\ \text{or}\ \frac{\pi}{2},\ \text{or}\ 0\ ;\qquad\text{(IV)}$$

" and since under the first two conditions the values of $a$ and $x$ are indefinitely near to each other

$$\frac{1}{\pi}\cdot\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=f(x),\ \text{or}\ \frac{f(x)}{2},\ \text{or}\ 0.\qquad\text{(V)}$$

" Extend this by integration from $p$ to $q$, then observing that " each half value, $\frac{f(x)}{2}$, occurs in two contiguous elements, except the " first of them and the last, we have

$$\frac{1}{\pi}\int_p^q\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=f(x),\ \text{or}\ \frac{f(x)}{2},\ \text{or}\ 0,\qquad\text{(VI)}$$

" according as $x$ lies within, upon, or without the limits $p$ and $q$."

This when read in an ordinary way seems to be faultless, but careful examination shows that it is not so.

Let us first examine (IV) without regard to how it was obtained, remembering that $\kappa$ is infinitesimally small compared with $da$.

It is plain that if $x$ and $a$ are not equal

$$\frac{\kappa da}{\kappa^2+(a-x)^2}=\frac{0.da}{0^2+(a-x)^2}=0,$$

and that if $x=a$

$$\frac{\kappa da}{\kappa^2+(a-x)^2}=\frac{\kappa da}{\kappa^2+0^2}=\frac{da}{\kappa}=\frac{da}{0}=\infty.$$

Consequently (IV) is not true, and therefore the process by which it was obtained must be faulty.

Let us now examine the process.

$$\Delta\tan^{-1}\frac{a-x}{\kappa}=\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa},$$

$$=\tan^{-1}\left(\frac{\frac{a+\Delta a-x}{\kappa}-\frac{a-x}{\kappa}}{1+\frac{a+\Delta a-x}{\kappa}\cdot\frac{a-x}{\kappa}}\right),$$

$$=\tan^{-1}\left(\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}\right).$$

Now $z$ and $\tan^{-1}z$ are interchangeable if both are infinitesimally small and of the same sign, but not in any other case. But if

$$z=\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

this is the case only when $x$ lies without the limits $a$ and $a+\Delta a$. If $x$ lie between the limits $z$ is infinitesimally small and negative, in which case putting

$$\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}\quad\text{for}\quad\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

is equivalent to putting $-0$ for $\pi$ (see the lines immediately following (I) above), and putting

$$\frac{\kappa\Delta a}{\kappa^2+(a-x)^2}\text{ instead of }\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

for $\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$, or $+0$ for $\pi$, makes it no better.

Lastly, when $x=a$ or $a+\Delta a$ the putting of

$$\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}\text{ for }\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$$

is equivalent to putting $\infty$ for $\frac{\pi}{2}$, and the putting of $\frac{\kappa\Delta a}{\kappa^2+(a-x)^2}$ instead of $\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$ for $\tan^{-1}\frac{\kappa\Delta a}{\kappa^2+(a+\Delta a-x)(a-x)}$, or $\infty$ for $\frac{\pi}{2}$, is no better.

Hence we conclude that we cannot substitute $\frac{\kappa\Delta a}{\kappa^2+(a-x)^2}$ for $\Delta\tan^{-1}\frac{a-x}{\kappa}$ except when $x$ lies without the limits $a$ and $a+\Delta a$. Therefore the process by which (IV) was obtained from (III), namely, the putting of $\frac{\kappa da}{\kappa^2+(a-x)^2}$ for $d\tan^{-1}\frac{a-x}{\kappa}$ is illegitimate.

Now, though (IV) is not true, (VI) is true. To prove it let us return to the beginning.

From (II) we have

$$
\begin{aligned}
f(x),\ \frac{f(x)}{2} \text{ or } 0 &= \frac{1}{\pi}f(x)\left\{\tan^{-1}\frac{a+\Delta a-x}{\kappa}-\tan^{-1}\frac{a-x}{\kappa}\right\},\\
&= \frac{1}{\pi}f(x)\int_a^{a+\Delta a}\frac{\kappa da}{\kappa^2+(a-x)^2},\\
&= \frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da f(a)}{\kappa^2+(a-x)^2},\\
&= \frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da f(a)}{\kappa^2+(a-x)^2},
\end{aligned}
$$

provided $\Delta a$ be infinitesimally small and $\kappa$ be infinitesimally small compared with $\Delta a$; because when $x$ lies between $a$ and $a+\Delta a$, or is equal to either of them, $f(a)$ will differ infinitesimally little from $f(x)$ while $a$ changes to $a+\Delta a$, and when $x$ lies without the limits $a$ and $a+\Delta a$ the substitution of $f(a)$ for $f(x)$ makes no difference because the other factor $\int_a^{a+\Delta a}\frac{\kappa da}{\kappa^2+(a-x)^2}=0$ in this case. Hence

$$
\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da. f(a)}{\kappa^2+(a-x)^2}=f(x),\ \frac{f(x)}{2},\text{ or } 0, \qquad \text{(VII)}
$$

according as $x$ lies within, upon, or without the limits $a$ and $a+\Delta a$. From this it follows that

$$\left.\begin{aligned}
&\frac{1}{\pi}\int_{x}^{x+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=\frac{f(x)}{2}\\
&\frac{1}{\pi}\int_{x-\Delta a}^{x}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=\frac{f(x)}{2}\\
&\frac{1}{\pi}\int_{x+m\Delta a}^{x+(m+1)\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=0
\end{aligned}\right\}\qquad\text{(VIII)}$$

where $m$ denotes any positive integer or any negative integer except $-1$.

(VII) and (VIII) are true on the supposition that $f(a)$ is continuous between the limits of integration which has been tacitly assumed.

If $x$ lie between the limits $p$ and $q$, and if $f(a)$ be continuous between these limits, it follows from (VIII) that

$$\frac{1}{\pi}\int_{p}^{q}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=\frac{1}{\pi}\left\{\int_{p}^{p+\Delta a}+\int_{p+\Delta a}^{p+2\Delta a}+\ldots\ldots+\int_{x-\Delta a}^{x}+\int_{x}^{x+\Delta a}+\right.$$

$$\left.\ldots\ldots+\int_{q-2\Delta a}^{q-\Delta a}+\int_{q-\Delta a}^{q}\right\}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2},$$

$$=0+0+\ldots\ldots+\frac{f(x)}{2}+\frac{f(x)}{2}+\ldots\ldots+0+0,$$

$$=f(x).$$

Similarly it may be shown that if $x$ is equal to $p$ or $q$, or lies without the limits $p$ and $q$, the result will be $\frac{f(x)}{2}$ or 0. Hence

$$\frac{1}{\pi}\int_{p}^{q}\frac{\kappa daf(a)}{\kappa^2+(a-x)^2}=f(x),\ \frac{f(x)}{2},\text{ or }0,\qquad\text{(IX)}$$

according as $x$ lies within, upon, or without the limits $p$ and $q$.

This is the same as (VI), which is thus proved to be true universally, provided $f(a)$ is continuous between the limits $p$ and $q$.

It could easily be shown that if $f(a)$ is discontinuous at a point $x$ between $p$ and $q$

$$\frac{1}{\pi}\int_{p}^{q}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=\frac{f(x-0)}{2}+\frac{f(x+0)}{2}.\qquad\text{(X)}$$

(IX) might also be written in this form, so that (X) is true universally whether $x$ be a point of continuity or discontinuity, if $x$ lie between $p$ and $q$.

Now (Todhunter's *Integral Calculus*, § 291)

$$\frac{\kappa}{\kappa^2+(a-x)^2}=\int_{0}^{\infty}\epsilon^{-\kappa b}\cos b(a-x)db.$$

Therefore (IX) becomes

$$\frac{1}{\pi}\int_p^q da f(a)\int_0^\infty \epsilon^{-\kappa b}\cos b(a-x)db = f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XI)}$$

according as $x$ lies within, upon, or without the limits $p$ and $q$.

There might be a little doubt as to the truth of (XI) for the reason that in arriving at it we have made $\Delta a$ infinitesimally small, that is, *we have put* $\Delta a = 0$ *before putting* $\kappa = 0$, *which is contrary to our initial supposition*, namely, that $\frac{a-x}{\kappa}$ and $\frac{a+\Delta a-x}{\kappa}$ are both infinitely great when $x$ lies between $a$ and $a+\Delta a$, and consequently that $\kappa$ is infinitesimally small compared with $\Delta a$. Glaisher in his paper signifies this by saying that $\kappa$ is an infinitesimal of a higher grade than $\Delta a$, and appears to think that this must be emphasised so as to remove any doubt as to Boole's result. Now, in whatever way the condition is expressed, it simply amounts to this, *that* $\kappa$ *should be put equal to* 0 *before* $\Delta a$ *is put equal to* 0. But in (XI) $\Delta a$ has *already* been put equal to 0, and $\kappa$ has *not yet* been put equal to 0, therefore *the initial condition has been violated.* If, therefore, we cannot justify this violation, we can place no dependence on (XI).

This justification might be attempted as follows:—

Going back to (VII), which is the foundation for the whole, it must be observed that it is *only an approximation.*

Now, if we put $x_1$ and $x_2$ for the values of $a$ which make $f(a)$ respectively the greatest and least as $a$ increases to $a+\Delta a$, then instead of (VII) we might write

$$\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2} < f(x_1) \text{ and } > f(x_2)$$

if $x$ lie between $a$ and $a+\Delta a$;

$$\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2} < \frac{f(x_1)}{2} \text{ and } > \frac{f(x_2)}{2} \qquad \text{(VII*)}$$

if $x$ equals either $a$ or $a+\Delta a$;

$$\frac{1}{\pi}\int_a^{a+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2} = 0$$

if $x$ lies without the limits $a$ and $a+\Delta a$.

Hence instead of (VIII) we might write

$$\left.\begin{array}{l}
\dfrac{1}{\pi}\displaystyle\int_{x}^{x+\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\frac{f(x_1)}{2}\text{ and }>\frac{f(x_2)}{2}\\[2ex]
\dfrac{1}{\pi}\displaystyle\int_{x-\Delta a}^{x}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\frac{f(x_1')}{2}\text{ and }>\frac{f(x_2')}{2}\\[2ex]
\dfrac{1}{\pi}\displaystyle\int_{x+m\Delta a}^{x+(m+1)\Delta a}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=0
\end{array}\right\}\qquad(\text{VIII}^*)$$

and going through the process by which we passed from (VIII) to (IX), we obtain instead of (IX)

$$\left.\begin{array}{l}
\dfrac{1}{\pi}\displaystyle\int_{p}^{q}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\frac{f(x_1)+f(x_1')}{2}\text{ and }>\frac{f(x_2)+f(x_2')}{2}\\
\text{if } x \text{ lies between } p \text{ and } q\,;\\[1ex]
\dfrac{1}{\pi}\displaystyle\int_{p}^{q}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}<\frac{f(x_1)}{2}\text{ or }\frac{f(x_1')}{2}\text{ and }>\frac{f(x_2)}{2}\text{ or }\frac{f(x_2')}{2}\\
\text{according as } x=p \text{ or } q\,;\\[1ex]
\dfrac{1}{\pi}\displaystyle\int_{p}^{q}\frac{\kappa da.f(a)}{\kappa^2+(a-x)^2}=0
\end{array}\right\}\qquad(\text{IX}^*)$$

if $x$ lies without the limits $p$ and $q$.

In all this it must be observed that $\Delta a$ is not equal to 0, and must not be confounded with $da$, for $da$ did not enter (IX*) as $\Delta a$ indefinitely diminished, but entered it in the passage from (II) to (VII*), in the latter of which are found both $da$ and $\Delta a$ as independent quantities. The danger of confounding these might have been avoided by writing (VII) thus

$$\frac{1}{\pi}\int_{a}^{a+\Delta a}\frac{\kappa dv.f(v)}{\kappa^2+(v-x)^2}=f(x),\ \frac{f(x)}{2},\text{ or }0$$

according as, &c.

Now, in (IX*) we may suppose $\Delta a$ to be as small as we please, provided we do not make it absolutely 0. But the smaller $\Delta a$ is made the more nearly will $f(x_1)$ and $f(x_2)$ be equal to each other and to $f(x)$ if $x=a$ or $a+\Delta a$ or lie between them. The same is true of $f(x_1')$ and $f(x_2')$. Therefore (IX*) can be made as nearly (IX) as we please by diminishing $\Delta a$ sufficiently, and therefore I *think* we may conclude that (IX) is true, and therefore that (XI) is true.

Assuming then that (XI) is true, and putting $p=\infty$ and $\;=-\infty$ we shall have

$$\frac{1}{\pi}\int_{-\infty}^{\infty} da f(a)\int_0^{\infty}\epsilon^{-\kappa b}\cos b(a-x)db = f(x) \qquad \text{(XII)}$$

for all real and finite values of $x$, with the same caution as before as in the case of points where $f(x)$ is discontinuous.

I shall now show that the order of operation in (XI) is reversible. To do this let us return to

$$\tan^{-1}\frac{a+\Delta a-x}{\kappa} - \tan^{-1}\frac{a-x}{\kappa} = \pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XIII)}$$

according as $x$ lies within, upon, or without the limits $a$ and $a+\Delta a$.

Now (Todhunter's *Integral Calculus*, § 285)

$$\tan^{-1}\left(\frac{r}{\kappa}\right) = \int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin rb}{b}db.$$

Therefore

$$\begin{aligned}\tan^{-1}\frac{a+\Delta a-x}{\kappa} - \tan^{-1}\frac{a-x}{\kappa} &= \int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin b(a+\Delta a-x)}{b}db \\ &\quad - \int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin b(a-x)}{b}db, \\ &= \int_0^{\infty}\epsilon^{-\kappa b}\frac{\sin b(a+\Delta a-x)-\sin b(a-x)}{b}db, \\ &= \int_0^{\infty}db\epsilon^{-\kappa b}\int_a^{a+\Delta a}\cos b(v-x)av.\end{aligned}$$

Therefore (XIII) becomes

$$\int_0^{\infty}db\epsilon^{-\kappa b}\int_a^{a+\Delta a}dv\cos b(v-x) = \pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XIV)}$$

according as $x$ lies within, upon, or without the limits $a$ and $a+\Delta a$. If we suppose $\Delta a$ to be exceedingly small $f(v)$ will vary exceedingly little while $v$ increases from $a$ to $a+\Delta a$, provided $f(v)$ is continuous between these limits. Therefore

$$\frac{1}{\pi}\int_0^{\infty}db.\epsilon^{-\kappa b}\int_a^{a+\Delta a}dv f(v)\cos b(v-x)$$

will differ exceedingly little from

$$\frac{f(a)}{\pi}\int_0^{\infty}db\epsilon^{-\kappa b}\int_a^{a+\Delta a}dv\cos b(v-x),$$

that is from $\quad f(a),\ \frac{f(a)}{2},\ \text{or } 0,$

according as $x$ lies within, upon, or without the limits $a$ and $a+\Delta a$.

Hence we have

$$\frac{1}{\pi}\int_0^{\infty} db\epsilon^{-\kappa b}\int_a^{a+\Delta a} dvf(v)\cos b(v-x)=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XV)}$$

according as $x$ lies between, upon, or without the limits $a$ and $a+\Delta a$.

We can now, in the same way as we passed from (VII) to (IX), pass from (XV) to

$$\frac{1}{\pi}\int_0^{\infty} db\epsilon^{-\kappa b}\int_p^{q} dvf(v)\cos b(v-x)=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XVI)}$$

according as $x$ lies within, upon, or without the limits $p$ and $q$; and this is the same as (XI) with the order of integration reversed.

The same doubt might be felt regarding the truth of (XVI) as regarding the truth of (XI), and may be removed in the same way.

If now $p$ and $q$ be put respectively equal to $-\infty$ and $+\infty$ we will have

$$\frac{1}{\pi}\int_0^{\infty} db.\epsilon^{-\kappa b}\int_{-\infty}^{\infty} dvf(v)\cos b(v-x)=f(x),\ \frac{f(x)}{2},\ \text{or } 0, \qquad \text{(XVII)}$$

where it must be observed that $p$ and $q$ are not to be put equal to $\infty$ until after both integrations, in those cases at least where $\int_{-\infty}^{\infty} dvf(v)\cos b(v-x)$ becomes infinite or indeterminate without the arbitrary introduction of such a multiplier as $\epsilon^{-\lambda v}$ under the integral (where $\lambda$ is put $=0$ after the integration). This I showed in my former paper to be true in the case where

$f(x)=t_1$ for positive values of $x$

and $=t_2$ „ negative „ „

in which case the integration with respect to $v$ between the limits $-\infty$ and $+\infty$ is indeterminate, and when rendered determinate by exponential multipliers the result comes out $\pm\frac{t_1-t_2}{2}$ according as $x$ is positive or negative (see end of my former paper). But if we put $p=-l$ and $q=+l$ we get

$$\frac{1}{\pi}\int_0^{\infty} db\epsilon^{-\kappa b}\int_{-l}^{l} dvf(v)\cos b(v-x)=\frac{1}{\pi}\Big[t_1.\tan^{-1}\frac{l-x}{\kappa}+t_2.\tan^{-1}\frac{l+x}{\kappa}$$
$$+(t_1-t_2)\tan^{-1}\frac{x}{\kappa}\Big]$$

which, on making $l$ infinite, becomes

$$\frac{1}{\pi}\left[t_1.\frac{\pi}{2}+t_2.\frac{\pi}{2}+(t_1-t_2)\tan^{-1}\frac{x}{\kappa}\right]$$

or $$\frac{t_1+t_2}{2}+\frac{t_1-t_2}{\pi}\tan^{-1}\frac{x}{\kappa};$$

and this gives $\frac{t_1+t_2}{2}+\frac{t_1-t_2}{\pi}.\frac{\pi}{2}=t_1$, when $x$ is positive,

and $$\frac{t_1+t_2}{2}+\frac{t_1-t_2}{\pi}\left(-\frac{\pi}{2}\right)=t_2 \text{ when } x \text{ is negative;}$$

which is as it should be.

Solving the same problem by (XI) we get

$$\frac{1}{\pi}\int_{-l}^{l}daf(a)\int_0^{\infty}\epsilon^{-\kappa b}\cos b(a-x)db$$

$$=\frac{1}{\pi}\left\{t_1\left(\tan^{-1}\frac{l-x}{\kappa}-\tan^{-1}\frac{-x}{\kappa}\right)+t_2\left(\tan^{-1}\frac{-x}{\kappa}-\tan^{-1}\frac{-l-x}{\kappa}\right)\right\},$$

which when $l$ is put $=\infty$ becomes

$$\frac{1}{\pi}\left\{t_1\left(\frac{\pi}{2}-\tan^{-1}\frac{-x}{\kappa}\right)+t_2\left(\tan^{-1}\frac{-x}{\kappa}+\frac{\pi}{2}\right)\right\},$$

or $$\frac{t_1+t_2}{2}-\frac{t_1-t_2}{\pi}\tan^{-1}\frac{-x}{\kappa};$$

which is also correct.

I had written a draft copy of this before I had the pleasure of seeing Mr Glaisher's paper. Dr Muir, who kindly lent it to me, had previously called my attention to the fact that my proposed demonstration was substantially the same as Glaisher's. I have, however, retained it in the present communication, as the procedure I adopt shows that it has the same foundation as Boole's proof—a point not brought out in Glaisher's paper.

Glaisher starting with

$$\int_0^{\infty}\frac{\sin rb}{b}db=\pm\frac{\pi}{2}, \qquad \text{(XVIII)}$$

according as $r$ is positive or negative, and therefore

$$\int_0^{\infty}\frac{\sin b(a+\Delta a-x)-\sin b(a-x)}{b}db=\pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XIX)}$$

according as $x$ lies within, upon, or without the limits $a$ and $a+\Delta a$, shows that this is equivalent to

$$\Delta a\int_0^\infty \cos b(a-x)db = \pi,\ \frac{\pi}{2},\ \text{or } 0, \qquad \text{(XX)}$$

according as $x$ is within, &c. Therefore

$$f(a).\Delta a\int_0^\infty \cos b(a-x)db = \pi f(a),\ \frac{\pi}{2}f(a), \text{or } 0, \qquad \text{(XXI)}$$

according as $x$ is within, &c. Hence we have

$$f(p).\Delta a\int_0^\infty \cos b(p-x)db + f(p+\Delta a).\Delta a\int_0^\infty \cos b(p+\Delta a-x)db$$

$$+ \ldots\ldots\ldots\ldots + f(p+n\Delta a)\Delta a\int_0^\infty \cos b(p+n\Delta a-x)\,db$$

$$= \pi f(x),\ \frac{\pi}{2}f(x), \text{or } 0, \qquad \text{(XXII)}$$

according as $x$ lies within, upon, or without the limits $p$ and $p+n\Delta a$; or putting $q$ for $p+n\Delta a$ and diminishing $\Delta a$ indefinitely this becomes

$$\int_p^q da f(a)\int_0^\infty \cos b(a-x)db = \pi.f(x),\ \frac{\pi}{2}f(x),\ \text{or } 0, \qquad \text{(XXIII)}$$

according as $x$ lies within, upon, or without the limits $p$ and $q$.

He now remarks that $\int_0^\infty \cos b(a-x)db$ is indeterminate and proposes to substitute for it the determinate form $\int_0^\infty \epsilon^{-\kappa b}\cos b(a-x)db$ to obtain which he proposes to start with $\int_0^\infty \epsilon^{-\kappa b}\frac{\sin rb}{b}db = \pm\frac{\pi}{2}$ instead of with $\int_0^\infty \frac{\sin rb}{b}db = \pm\frac{\pi}{2}$. But then it must be noted that

$$\int_0^\infty \epsilon^{-\kappa b}\frac{\sin rb}{b}db = \tan^{-1}\frac{r}{k}.$$

So that Glaisher's second departure is the same as Boole's.

If Glaisher's result (XXIII) be written in its equivalent form

$$\int_0^\infty db\int_p^q da f(a)\cos b(a-x) = \pi f(x), \frac{\pi}{2}f(x),\ \text{or } 0, \qquad \text{(XXIV)}$$

it is no longer indeterminate; and if in this for $p$ and $q$ we substitute $-l$ and $+l$ and increase $l$ indefinitely, we obtain

$$\mathrm{L}_{l=\infty}\int_0^\infty db\int_{-l}^{l} da f(a)\cos b(a-x) = \pi f(x) \qquad \text{(XXV)}$$

which gives correct results in every example I have tested it by *provided $l$ be not put $=\infty$ till after both integrations.* It has, moreover, the advantage of being more easily handled than any of the other forms.

This same form has been proved by G(regory?) by a symbolical method, but owing to his using vague considerations of grades of infinity he has obtained $\int_0^\infty db\int_0^\infty da f(a)\cos b(a-x)=\frac{\pi}{2}f(x)$ instead of $\pi f(x)$. Besides he makes no restriction as to *when* the superior limit of $a$ is to be put $=\infty$, without which the theorem is unsatisfactory in its application to examples. I have succeeded in modifying G(regory?)'s proof so as to avoid doubtful considerations of grades of infinity and zero, and have obtained the proper value $\pi f(x)$, and the nature of the investigation points out that $l$ should not be put $=\infty$ till after both integrations. As however my paper is already too long I shall not give this investigation.

Equation (XXV) also follows from (XVI) by putting $\kappa=0$ *before the integrations*; and this is lawful, because (XVI) is derived from (XIV), which is true if $\kappa$ be put equal to 0 before integrating. For

$$\int_0^\infty db\int_a^{a+\Delta a}dv\cos b(v-x)=\int_0^\infty db\left\{\frac{\sin(a+\Delta a-x)b-\sin(a-x)b}{b}\right\},$$

$$=\int_0^\infty db\frac{\sin b(a+\Delta a-x)}{b}-\int_0^\infty db\frac{\sin b(a-x)}{b},$$

$$=\frac{\pi}{2}-\left(-\frac{\pi}{2}\right),\ \frac{\pi}{2}-0\text{ or }0-\left(-\frac{\pi}{2}\right),$$

$$\frac{\pi}{2}-\frac{\pi}{2}\text{ or }-\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)$$

$$=\pi,\ \frac{\pi}{2},\text{ or }0,$$

according as $x$ lies within, upon, or without the limits $a$ and $a+\Delta a$. We can thus avoid the doubtful proceeding of putting $\Delta a=0$ before putting $\kappa=0$, which is a violation of the initial conditions.

After objecting to the arbitrary introduction of such a factor as $\epsilon$—$\lambda v$ to make such an integral as $\int_0^\infty\cos b(v-x)dv$ determinate I ought to justify my own *seeming* use of this method in my former paper.

Taking equation (A) of said paper, which I observe is wrongly stated there,

$$\phi(x)=\frac{1}{2l}\int_{-l}^{+l}\phi(v)\,dv+\frac{1}{\pi}\Sigma\left\{\int_{-l}^{+l}\cos w(x-v).\phi(v)dv\Delta w\right\},\qquad (A)$$

should the integral in the second term of the right hand member become indeterminate when $l$ is put $=\infty$, I suppose $\epsilon^{\mp\lambda x}\phi(x)$ to be put for $\phi(x)$ according as $x$ is positive or negative, when (A) becomes

$$\epsilon^{\mp\lambda x}\phi(x)=\frac{1}{2l}\int_{-l}^{+l}\epsilon^{\mp\lambda v}\phi(v)\,dv+\frac{1}{\pi}\Sigma\left\{\int_{-l}^{+l}\cos w(x-v).\epsilon^{\mp\lambda v}\phi(v)dv\Delta w\right\},\qquad (A^*)$$

which will now remain determinate when $l$ becomes infinite, and, after the integrations are performed, we can obtain $\phi(x)$ by putting $\lambda=0$ in the result.

In justice to De Morgan I ought to notice that he has given Poisson's formula in the same order as (XII) in which there is no danger of putting $l=\infty$ too soon. But he does not say at what stage $\kappa$ is to be put equal to 0. His words, if they indicate anything as to this point, seem to me to say that this is to be done before any of the integrations are effected. Besides this, he does not seem to see that his form labours under the same fault, if fault it be, with which he charges the verifications he discusses, namely, the order of integration is inverted.

## Theorems on three mutually tangent circles.

By Thomas Muir, M.A., LL.D.

This paper will appear later.

Mr William Peddie gave some notes on Reflected Rainbows, in which the bow and its reflection due to the image of the sun were discussed in relation to the ordinary bow and its reflection.

# Radical Axes in Spherical Geometry.

### By R. E. Allardice, M.A.

It is evident that by an extension of the method of deriving from any triangle its polar triangle, it is possible to derive from any figure whatever another figure, the properties of which may be deduced at once from those of the first. This may be done either by imagining a point to move along the original figure and considering the envelope of the great circle of which the moving point is the pole; or by imagining a great circle to envelope the figure and considering the locus of its pole. In both cases two figures will be obtained; but these will be antipodal, and will therefore have like properties. Since the point of intersection of two great circular arcs is the pole of the great circular arc which joins the poles of these two arcs, the two methods of derivation mentioned above will lead to the same derived figure. These methods of transformation of figures are evidently closely analogous to that of reciprocal polars.

The ordinary polar triangle may be obtained as the locus of the pole of a great circular arc which is made to revolve round the original triangle; and the fact that this arc revolves through the exterior angle of the one triangle, while its pole moves along a side of the other, shows that the angles of the one triangle are the supplements of the sides of the other; and that a great circular arc, cutting an interior angle of the one, corresponds to an external point in a side of the other.

A point may, of course, be considered as enveloped by all the arcs that pass through it, just as an arc is traced out by all the points that lie in it.

It will be well to notice the following points connected with the transformation of circles and related points and lines, as bearing on the subject of this note:—

A circle transforms into a circle; its centre into the great circle parallel to (or concentric with) the derived circle.

A tangent transforms into a point on the circumference; the point of contact into the tangent at the point into which the tangent transforms. A secant transforms into a point on the sphere; the points of section into the tangents from that point.

If two circles be now considered, it will be seen that the central axis transforms into either of the (antipodal) points of intersection of the two great circles into which the two centres transform. Again, since a point of intersection is a point common to two circles, and to this corresponds a tangent common to two circles; it would seem that points of intersection and common tangents transform into one another. But two circles can only intersect in two points, while in a plane at least two circles may have four common tangents; and a little examination will show that this may also be the case on the surface of the sphere. Now, on examination of the figure it will be seen that if a great circular arc be made to revolve in the same direction round each of the circles, either of its poles will be in the same direction for the two circles when it is a direct common tangent; and in a different direction when it is an inverse common tangent. Hence to the two direct common tangents correspond the points of intersection; and to the inverse common tangents, antipodal points on the two circles. From this it follows that there can be only two pairs of such points, except when the circles are antipodal, in which case there are an infinite number, and likewise an infinite number of common tangents. (It is evident that a pair of antipodal circles transforms into a pair of antipodal circles).

The point of intersection of the direct common tangents (the external centre of similitude) transforms into the arc joining the points of intersection, that is, the radical axis; and since the great circular arc which passes through the two pairs of antipodal points corresponds to the internal centre of similitude, it may conveniently be called the second radical axis of the two circles.

If the potency of a point with respect to a circle be defined as $\tan\frac{1}{2}a\tan\frac{1}{2}b$, where $a$ and $b$ are the segments of a secant through the point, the potencies of any point on the first radical axis with respect to two circles will be the same, while those of any point on the second radical axis will be the reciprocals of one another. From this it follows at once that the six radical axis of three circles pass in sets of three through four points.

Corresponding to the potency of a point with regard to a circle, there is the following property of a line and a circle, suggesting a "potency of a line" with regard to a circle. If tangents be drawn from any point P in a fixed line PQ to a given circle, touching it at A and B, then $\tan\frac{1}{2}\mathrm{APQ}\cot\frac{1}{2}\mathrm{BPQ}$ will be constant. This property follows from the fact that it is the polar theorem to that which gives the corresponding property of a point and a circle; and the quantity $\tan\frac{1}{2}APQ\cot\frac{1}{2}BPQ$ may be called the potency of the line PQ with reference to the circle. The above theorem is also true *in plano*, and affords a simple proof that the six centres of similitude of three circles lie in sets of three on four straight lines. To the fact of the collinearity of the centres of two circles and their two centres of similitude corresponds the proposition that the two parallel great circles and the two radical axes are concurrent.

The following examples of properties of circles which correspond according to the above method of transformation may also be noted.

To the fact that the radius of a circle is of constant length, corresponds the fact that the angle which the tangent to a circle makes with the parallel great circle is constant; and the proposition that the tangents make equal angles with the chord of contact transforms into the proposition that the two tangents from the same point are equal.

Again, the centres of similitude of two circles divide the line joining the centres of the circles so that the ratio of the sines of the segments is equal to the ratio of the sines of the radii. This theorem gives on transformation the theorem that the radical axes divide the angle between the parallel circles so that the ratio of the sines of the parts is equal to the ratio of the sines of the angles which the tangents to the circles make with the parallel great circles. This theorem evidently cannot have any analogue in plane geometry, as must in general be the case with the polars of such theorems as refer to the centre of a circle, since the line in a plane corresponding to a parallel great circle is altogether at an infinite distance.

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## *Fifth Meeting, March 13th 1885.*

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GEORGE THOM, Esq., M.A., Vice-President, in the Chair.

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## Gilbert's Method of Treating Tangents to Confocal Conicoids.

### By GEORGE A. GIBSON, M.A.

The method of treating tangents to confocal conicoids, of which I propose to give an account, is discussed in the number for December

1867 of the *Nouvelles Annales de Mathématiques.* The writer of the paper referred to is M. Ph. Gilbert, Professor at the University of Louvain. The results arrived at are in nearly every case already well known, but the method of reaching them is somewhat novel, and I have thought that it might interest the members of this society if I were to give a statement of the chief methods and results of Gilbert's paper. In one or two cases I have altered the proofs, and I have added two or three propositions that seemed to follow naturally from the equations dealt with. Gilbert deals only with central conicoids; but I have put in an equation for the paraboloids that corresponds to Gilbert's fundamental one.

Gilbert's method of dealing with tangents to confocal conicoids is based upon a certain expression for the angle between the normals at two points, one on each of two confocals. Let the confocals be given by the equation

$$\frac{x^2}{\lambda^2}+\frac{y^2}{\lambda^2-b^2}+\frac{z^2}{\lambda^2-c^2}=1$$

$2\lambda$ being the primary axis of any one of the series. Any individual confocal may be referred to by its primary semi-axis.

Let M be a point on $\lambda$, its co-ordinates being $(xyz)$,
N " $\theta$, " " $(\xi\eta\zeta)$,
$p_\lambda$ the perpendicular from centre on tangent plane to $\lambda$ at M,
$p_\theta$ " " " $\theta$ at N,
$\delta$ the distance between M and N.

Let $(\delta\lambda)$, $(\delta\theta)$ be the angles between MN and the normals to $\lambda$ and $\theta$ at M and N respectively, and $(\theta\lambda)$ the angle between the normals themselves, the normals being in all cases drawn *outwards.*

The direction cosines of normal at M will be $\left\{\frac{p_\lambda x}{\lambda^2}, \frac{p_\lambda y}{\lambda^2-b^2}, \frac{p_\lambda z}{\lambda^2-c^2}\right\}$;

" " N " $\left\{\frac{p_\theta \xi}{\theta^2}, \frac{p_\theta \eta}{\theta^2-b^2}, \frac{p_\theta \zeta}{\theta^2-c^2}\right\}$;

$$\cos(\theta\lambda)=p_\lambda\,.\,p_\theta\left\{\frac{x\xi}{\lambda^2\theta^2}+\frac{y\eta}{(\lambda^2-b^2)(\theta^2-b^2)}+\frac{z\zeta}{(\lambda^2-c^2)(\theta^2-c^2)}\right\};$$

$$\cos(\delta\lambda)=\frac{p_\lambda}{\delta}\left\{\frac{x(\xi-x)}{\lambda^2}+\frac{y(\eta-y)}{\lambda^2-b^2}+\frac{z(\zeta-z)}{\lambda^2-c^2}\right\}$$

$$=\frac{p_\lambda}{\delta}\left\{\frac{x\xi}{\lambda^2}+\frac{y\eta}{\lambda^2-b^2}+\frac{z\zeta}{\lambda^2-c^2}-1\right\};\ \text{since } xyz \text{ lies on } \lambda;$$

$$\therefore\quad \frac{\delta.\cos(\delta\lambda)}{p_\lambda}=\frac{x\xi}{\lambda^2}+\frac{y\eta}{\lambda^2-b^2}+\frac{z\zeta}{\lambda^2-c^2}-1.$$

Similarly, $\dfrac{\delta\cos(\delta\theta)}{p_\theta} = \dfrac{x\xi}{\theta^2} + \dfrac{y\eta}{\theta^2 - b^2} + \dfrac{z\zeta}{\theta^2 - c^2} - 1.$

$$\therefore \quad \frac{\delta.\cos(\delta\lambda)}{p_\lambda} - \frac{\delta.\cos(\delta\theta)}{p_\theta} = (\theta^2 - \lambda^2)\left\{\frac{x\xi}{\lambda^2\theta^2} + \frac{y\eta}{(\lambda^2 - b^2)(\theta^2 - b^2)} + \frac{z\zeta}{(\lambda^2 - c^2)(\theta^2 - c^2)}\right\}$$

$$= (\theta^2 - \lambda^2)\frac{\cos(\theta\lambda)}{p_\lambda \cdot p_\theta};$$

$$\therefore \quad \cos(\theta\lambda) = \frac{\delta}{\theta^2 - \lambda^2}\left\{p_\theta\cos(\delta\lambda) - p_\lambda\cos(\delta\theta)\right\} \ldots\ldots(\mathrm{A})$$

This is Gilbert's fundamental equation, and from it he deduces very readily many theorems on confocal conicoids.

1. Suppose M and N to coincide, *i.e.*, let $\delta$ be zero, then cos $(\theta\lambda)$ will be zero. Hence :—"Two confocal conicoids through the same point cut each other orthogonally."

2. If $\cos(\delta\lambda)$ and $\cos(\delta\theta)$ be each zero, *i.e.*, if the line MN touch both the conicoids, then $\cos(\theta\lambda)$ will still be zero. Hence :—"If two confocals touch the same straight line, and if we draw the tangent planes at the points of contact, these planes will cut at right angles." Or, as the theorem may be stated :—"If two confocals be viewed from any point, their apparent contours cut at right angles wherever they appear to intersect."

3. Suppose MN to touch $\theta$ at N ; then $\cos(\delta\theta)$ vanishes, and

$$\cos(\theta\lambda) = \frac{\delta p_\theta}{\theta^2 - \lambda^2} \cdot \cos(\delta\lambda) \ldots\ldots(\mathrm{B})$$

This equation is really that which Gilbert first proves ; but it is evidently included in equation (A).

4. If $(\lambda\mu\nu)$ be the primary semi-axes of the three confocals through M, and if MN touch $\theta$ at N, then by equation (B) we know that $\cos(\theta\lambda) = \dfrac{\delta p_\theta}{\theta^2 - \lambda^2} \cdot \cos(\delta\lambda)$ ; $\cos(\theta\mu) = \dfrac{\delta p_\theta}{\theta^2 - \mu^2} \cdot \cos(\delta\mu)$ and $\cos(\theta\nu) = \dfrac{\delta p_\theta}{\theta^2 - \nu^2} \cdot \cos(\delta\nu).$

Now, since the normals to the three confocals through M are mutually perpendicular, these may be taken as rectangular axes. With reference to these axes the direction cosines of MN are $\cos(\delta\lambda)$, $\cos(\delta\mu)$, $\cos(\delta\nu)$, and of the normal at N $\cos(\theta\lambda)$, $\cos(\theta\mu)$, $\cos(\theta\nu)$;

$$\therefore \quad \cos(\delta\theta) = \cos(\delta\lambda) \,.\, \cos(\theta\lambda) + \cos(\delta\mu) \,.\, \cos(\theta\mu) + \cos(\delta\nu) \,.\, \cos(\theta\nu)$$
$$= 0, \text{ since MN touches } \theta.$$

Putting in the values just given for $\cos(\theta\lambda)$, $\cos(\theta\mu)$, $\cos(\theta\nu)$, we get

$$\frac{\cos^2(\delta\lambda)}{\theta^2 - \lambda^2} + \frac{\cos^2(\delta\mu)}{\theta^2 - \mu^2} + \frac{\cos^2(\delta\nu)}{\theta^2 - \nu^2} = 0.$$

As every tangent line from M to conicoid $\theta$ fulfils this condition, the equation of the enveloping cone must be

$$\frac{x^2}{\theta^2 - \lambda^2} + \frac{y^2}{\theta^2 - \mu^2} + \frac{z^2}{\theta^2 - \nu^2} = 0.$$

The form of the equation shows that the normals to the three confocals are the principal axes of the cone.

5. If $\theta_1$, $\theta_2$ be the primary semiaxes of the two conicoids which touch a line through a point M, the direction cosines of which, referred to the three normals at M, are $l$, $m$, $n$, then the following will, by § 4, be an identical equation :—

$$l^2(\theta^2 - \mu^2)(\theta^2 - \nu^2) + m^2(\theta^2 - \nu^2)(\theta^2 - \lambda^2) + n^2(\theta^2 - \lambda^2)(\theta^2 - \mu^2)$$
$$= (\theta^2 - \theta_1^2)(\theta^2 - \theta_2^2)$$

$$\therefore \quad l^2 = \frac{(\lambda^2 - \theta_1^2)(\lambda^2 - \theta_2^2)}{(\lambda^2 - \mu^2)(\lambda^2 - \nu^2)}, \text{ with similar values for } m \text{ and } n.$$

Now, as a particular case, let $\mu = \nu = b$ (and $\therefore \theta_2 = b$), *i.e.*, take M on the focal hyperbola. We thus get $l^2 = \dfrac{\lambda^2 - \theta_1^2}{\lambda^2 - b^2}$. Hence the theorem— :

"The enveloping cone of an ellipsoid is one of revolution, when its vertex lies on the focal hyperbola, the axis of the cone being the tangent of the hyperbola at the point." The vertical angle of the cone is $2 \tan^{-1} \sqrt{\left(\dfrac{\theta_1^2 - b^2}{\lambda^2 - \theta_1^2}\right)}$.

6. By equation (B) $\dfrac{\cos^2(\theta\lambda)}{p_\theta^{\,2}\delta^2} = \dfrac{\cos^2(\delta\lambda)}{(\theta^2 - \lambda^2)^2}$

and $\cos^2(\theta\lambda) + \cos^2(\theta\mu) + \cos^2(\theta\nu) = 1$

$$\therefore \quad \frac{1}{p_\theta^{\,2}\delta^2} = \frac{\cos^2(\delta\lambda)}{(\lambda^2 - \theta^2)^2} + \frac{\cos^2(\delta\mu)}{(\mu^2 - \theta^2)^2} + \frac{\cos^2(\delta\nu)}{(\nu^2 - \theta^2)^2}$$

Suppose $\mu = \nu = b$, and we get, after substituting the values for the direction cosines given in the last article,

$$p_\theta \delta = \sqrt{(\lambda^2 - \theta_1^2)(\theta_1^2 - b^2)}.$$

In words—"The rectangle contained by the side of a cone of revolution enveloping an ellipsoid, intercepted between the vertex and point of contact, and the perpendicular from the centre on the tangent plane at that point is constant."

7. We may write the value for $\cos^2(\delta\lambda)$ in the form

$\dfrac{\theta^2 - \lambda^2}{p_\theta \delta} \,.\, \cos(\delta\lambda) \,.\, \cos(\theta\lambda)$, and we therefore find that

$$(\theta^2 - \lambda^2)\cos(\delta\lambda) \,.\, \cos(\theta\lambda) + \&c. \quad = p_\theta \delta$$

that is, $\lambda^2\cos(\delta\lambda) \,.\, \cos(\theta\lambda) + \mu^2\cos(\delta\mu) \,.\, \cos(\theta\mu) + \nu^2\cos(\delta\nu) \,.\, \cos(\theta\nu)$

$$= -p_\theta \delta$$

In words—"If from a point M we draw a tangent to a conicoid meeting it in N, and if we take on the normals to the three confocals through M lengths equal to the respective primary semiaxes of these confocals, the sum of the products of the projections of these lengths on the tangent by their projections on the normal to the given conicoid at N is equal to the product of the distance between M and the point of contact N by the perpendicular from the centre on the tangent plane at N."

8. Still using equation (B) we see that

$$(\theta^2 - \lambda^2)\cos^2(\theta\lambda) = p_\theta \delta \,.\, \cos(\delta\lambda) \,.\, \cos(\theta\lambda).$$

$$\begin{aligned} \therefore \quad & (\theta^2 - \lambda^2)\cos^2(\theta\lambda) + (\theta^2 - \mu^2)\cos^2(\theta\mu) + (\theta^2 - \nu^2)\cos^2(\theta\nu) \\ & = \delta p_\theta \{\cos(\delta\lambda) \,.\, \cos(\theta\lambda) + \cos(\delta\mu) \,.\, \cos(\theta\mu) + \cos(\delta\nu)\cos(\theta\nu)\} \\ & = 0. \end{aligned}$$

$$\therefore \quad \lambda^2\cos^2(\theta\lambda) + \mu^2\cos^2(\theta\mu) + \nu^2\cos^2(\theta\nu) = \theta^2.$$

This equation proves a theorem of Chasles, viz :—If on a fixed plane P we take any point M and measure on the normals to the three confocals through M lengths equal to their respective primary semiaxes, the sum of the squares of the projections of these lengths on the normal to the plane P will be constant for all positions of M on the plane. This equation also gives the primary axis of the conicoid which touches a given plane, as the equation of § 4 gives the axes of the conicoids which touch a given line.

9. Suppose a normal drawn from M $(\lambda,\mu,\nu)$ to a conicoid $\lambda'$ meeting it in N. Denote by $\mu'$, $\nu'$ the primary semi-axes of the other two confocals through N. Then since the normal to $\lambda'$ is a tangent to $\mu'$ and $\nu'$ the following equation will connect the direction cosines of the normal :—

$$\frac{\cos^2(\lambda'\lambda)}{(\mu'^2-\lambda^2)(\nu'^2-\lambda^2)}+\frac{\cos^2(\lambda'\mu)}{(\mu'^2-\mu^2)(\nu'^2-\mu^2)}+\frac{\cos^2(\lambda'\nu)}{(\mu'^2-\nu^2)(\nu'^2-\nu^2)}=0.$$

10. Returning to equation (A), we may suppose M and N to lie on the same conicoid, *i.e.*, take $\theta=\lambda$ and $\therefore\ \dfrac{p_\lambda}{\cos(\delta\lambda)}=\dfrac{p_\theta}{\cos(\delta\theta)}$ where $p_\lambda, p_\theta$ denote the perpendiculars from the centre on the tangent planes at M, N respectively. If a plane be drawn through the centre parallel to the tangent plane at M meeting MN in M′, then MM′ will be equal to $\dfrac{p_\lambda}{\cos(\delta\lambda)}$. Hence the theorem :—"If we take any chord MN of a conicoid, and through the centre draw planes parallel to the tangent planes at M, N meeting the chord in M′, N′ respectively, MM′ will be equal to NN′." Also, it may readily be deduced from this theorem that "The plane through the centre and the line of intersection of the tangent planes at M, N will bisect M′N′."

11. If in equation (A) $\cos(\theta\lambda)=0$, the equation $\dfrac{p_\lambda}{\cos(\delta\lambda)}=\dfrac{p_\theta}{\cos(\delta\theta)}$ still holds.

In this case the interpretation of the equation is :—"If on two confocal conicoids two points M, N be taken such that the normals at these points are perpendicular to each other, and if through the common centre planes be drawn parallel to the tangent planes at M, N cutting MN in M′,N′ respectively, then MM′ will be equal to NN′." From this it readily follows that, "The plane through the centre and the line of intersection of the tangent planes at M, N will bisect MN."

12. By § 5, we know that $\cos^2(\delta\lambda)=\dfrac{(\lambda^2-\theta_1^2)(\lambda^2-\theta_2^2)}{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}$ where $\lambda\mu\nu$ are the primary semi-axes of the three confocals through a point M, and $\theta_1\ \theta_2$ those of the two confocals touched by a line through M. If then a plane be drawn through the centre parallel to the tangent

plane at M, the square of the part intercepted on the line between M and this plane through the centre will be $p_\lambda{}^2 \dfrac{(\lambda^2-\mu^2)(\lambda^2-\nu^2)}{(\lambda^2-\theta_1{}^2)(\lambda^2-\theta_2{}^2)}$ that is, $\dfrac{\lambda^2(\lambda^2-b^2)(\lambda^2-c^2)}{(\lambda^2-\theta_1{}^2)(\lambda^2-\theta_2{}^2)}$, a value constant for all points on $\lambda$. Thus, "If from any point of a central conicoid a line be drawn touching two given confocals, the portion of this line intercepted between the point and the plane through the centre parallel to the tangent plane at the point will be constant." As a particular case, suppose the line to pass through the two focal conics and it is clear that "The part intercepted on a bifocal chord through a point M on a conicoid and the plane through the centre parallel to the tangent plane at M is equal to the primary semi-axis of the conicoid."

13. Equation (A) may be written in the form

$$\theta^2-\lambda^2=p_\theta.\frac{\delta\cos(\delta\lambda)}{\cos(\theta\lambda)}-p_\lambda.\frac{\delta\cos(\delta\theta)}{\cos(\theta\lambda)}\ldots\ldots(\mathrm{C})$$

Now if the normal at N meet the tangent plane at M in P, it is easily proved that $\mathrm{NP}=\dfrac{\delta\cos(\delta\lambda)}{\cos(\theta\lambda)}$. Denote this intercept by $\omega_\theta$, which will be positive or negative according as the normal is drawn *inwards* or *outwards* to meet the tangent plane. If $\omega_\lambda$ denote $\dfrac{\delta\cos(\delta\theta)}{\cos(\theta\lambda)}$ with the same convention as to sign, (C) may be written $\theta^2-\lambda^2=p_\theta\omega_\theta-p_\lambda\omega_\lambda$. Hence:—"If at any two points M, N on two confocal conicoids the normals be drawn, the products of the intercepts on these normals between the points of contact and the tangent planes by the perpendiculars from the centre on these tangent planes differ by a constant quantity; the constant being the difference of the squares of the primary semi-axes of the confocals."

As a particular case, suppose the tangent planes at M and N parallel; therefore $\omega_\theta=-\omega_\lambda=p_\theta-p_\lambda$ and $\therefore\ p_\theta{}^2-p_\lambda{}^2=\theta^2-\lambda^2$. Thus:—"If two parallel tangent planes be drawn to two confocals, the difference of the squares of the perpendiculars from the centre on these planes will be constant."

14. If $\lambda=\theta$, that is, if M and N be on the same conicoid, $p_\theta\omega_\theta=p_\lambda\omega_\lambda$. Therefore:—"If normals be drawn at two points M, N on a conicoid the products of the intercepts on these normals between the surface

and the tangent planes at N, M by the perpendiculars from the centre on the tangent planes at M, N are equal to each other."

15. Again, should the line MN touch the conicoid $\theta$, $\omega_\lambda$ would be zero, and the equation would become $p_\theta \omega_\theta = \theta^2 - \lambda^2$. Hence:—"If from any point M on a conicoid a tangent be drawn meeting a confocal in N, the product of the perpendicular from the centre on the tangent plane at N by the intercept on the normal at N between the tangent planes at M and N is constant; the constant being the difference of the squares of the primary semi-axis of the confocals."

It is easy to get an equation analogous to (A) for confocal paraboloids. Let the series of paraboloids be given by

$$\frac{y^2}{b+\lambda} + \frac{z^2}{c+\lambda} = 2x + \lambda.$$

As before,

Let M be a point on $\lambda$, its co-ordinates being $(xyz)$,
N „ $\theta$, „ „ $(\xi\eta\zeta)$,
$p_\lambda$ perpendicular from $(-x,0,0)$ on tangent plane to $\lambda$ at M,
$p_\theta$ „ $(-\xi,0,0)$ „ $\theta$ at N,
$\delta$ the distance between M, N.

Proceeding to find the values for $\cos(\delta\lambda)$, $\cos(\delta\theta)$, $\cos(\theta\lambda)$ we get

$$\frac{\lambda.\delta\cos(\delta\lambda)}{p_\lambda} = \left\{\lambda + \xi + x - \frac{\eta y}{b+\lambda} - \frac{\zeta z}{c+\lambda}\right\},$$

$$\frac{\theta.\delta\cos(\delta\theta)}{p_\theta} = \left\{\theta + \xi + x - \frac{\eta y}{b+\theta} - \frac{\zeta z}{c+\theta}\right\},$$

$$\cos(\theta\lambda) = \frac{p_\lambda p_\theta}{\lambda\theta}\left\{1 + \frac{\eta y}{(b+\theta)(b+\lambda)} + \frac{\zeta z}{(c+\theta)(c+\lambda)}\right\},$$

$$\cos(\theta\lambda) = \frac{\delta}{\lambda-\theta}\left\{\frac{p_\theta}{\theta}\cos\delta\lambda - \frac{p_\lambda}{\lambda}\cos\delta\theta\right\}.$$

Suppose $\delta = 0$; $\therefore \cos(\theta\lambda) = 0$. Hence confocal paraboloids intersect orthogonally.

Suppose $\lambda\mu\nu$ to be the parameters of the three confocals through a given point M, $\theta$ that of another confocal, and we get as before for the equation of the cone referred to the three normals:—

$$\frac{x^2}{\lambda-\theta} + \frac{y^2}{\mu-\theta} + \frac{z^2}{\nu-\theta} = 0.$$

The following cases in confocal conics may be noted.

The equation to the two tangents from a point M (whose co-ordinates are $\lambda\mu$) to a conic $\theta$ is, since the equation

$$\cos(\theta\lambda) = \frac{\delta}{\lambda^2 - \theta^2}\left\{p_\lambda\cos(\delta\theta) - p_\theta\cos(\delta\lambda)\right\} \text{ holds,}$$

$$\frac{\cos^2\delta\lambda}{\lambda^2 - \theta^2} + \frac{\cos^2\delta\mu}{\mu^2 - \theta^2} = 0,\ \lambda > \mu.$$

Let $\phi$ denote the angle between one of these tangents and the tangent to the confocal ellipse through M,

$$\therefore\ \sin^2\phi = \cos^2(\delta\lambda),\ \cos^2\phi = \cos^2(\delta\mu)\ ;$$

$$\frac{\sin^2\phi}{\lambda^2 - \theta^2} = \frac{\cos^2\phi}{\theta^2 - \mu^2} = \frac{1}{\lambda^2 - \mu^2}.$$

Hence:—The two tangents to an ellipse from any point are equally inclined to the tangent to a confocal ellipse through the point.

We also have the value of $\theta^2$ given by $\lambda^2\cos^2\phi + \mu^2\sin^2\phi = \theta^2$ which is the corresponding theorem to that of No. 8.

Corresponding to No. 10 is the theorem:—If MN be any chord of a conic, and if lines drawn through the centre parallel to the tangents at M and N meet the chord at M′ and N′ respectively, MM′ will be equal to NN′; and the line through the centre and the point of intersection of the tangents will bisect M′N′.

No. 11 gives the theorem:—If on two confocal conics two points M, N be taken such that the normals at these points are perpendicular to each other, and if through the common centre lines be drawn parallel to the tangents at M, N cutting MN in M′, N′ respectively, then MM′ will be equal to NN′; and the line joining the centre to the point of intersection of the tangents will bisect M′N′.

The theorem corresponding to that of No. 12 may be stated:—If from a point M on a central conic a tangent be drawn to a confocal conic, the intercept on the tangent between M and a central radius parallel to the tangent at M is constant. This holds when the confocal becomes the line-ellipse.

From No. 13 we get the theorem:—If at any two points MN on two confocal conics the normals be drawn, the products of the intercepts on these normals between the points of contact and the tangents by the perpendiculars from the centre on these tangents differ by a constant quantity—the constant being the difference of the squares of the primary semi-axes.

If the tangents at M and N be parallel, the squares of the perpendiculars from the centre on these tangents differ by a constant.

From No. 15 we get the theorem:—If from any point M on a conic, a tangent be drawn meeting a confocal in N, the product of the perpendicular from the centre on the tangent at N by the intercept on the normal at N between the tangents at M and N is constant.

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Mr J. S. Mackay gave a synopsis of Frans Schooten's "Geometry of the Rule," as it is contained in the second book of the *Exercitationes Mathematicae*, Leyden, 1657.

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Mr P. Alexander contributed a note on the two definite integrals

$$\int_0^\infty \sin nx dx \quad \text{and} \quad \int_0^\infty \cos nx dx.$$

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*Sixth Meeting, April 10th,* 1885.

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A. J. G. Barclay, Esq., M.A., President, in the Chair.

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## Note on the evaluation of functions of the Form $0^0$.

By T. B. Sprague, M.A., F.R.S.E.

Let $f(t)$, $\phi(t)$, be two functions of $t$, such that they both vanish with $t$, that is, $f(0) = 0$, $\phi(0) = 0$; and put

$$z = \{f(t)\}^{\Phi t}.$$

Then, in order to find the limiting value of $z$ when $t = 0$, we proceed as follows:—

$$\mathrm{Log} z = \phi(t) \, . \log f(t) = \frac{\log f(t)}{\dfrac{1}{\phi(t)}}$$

This fraction takes the form $-\frac{\infty}{\infty}$ when $t = 0$, and we therefore have

$$\mathrm{Lt}\log z = \mathrm{Lt}\frac{\log[f(t)]}{\dfrac{1}{\phi(t)}} = \mathrm{Lt}\frac{\dfrac{f'(t)}{f(t)}}{-\dfrac{\phi'(t)}{\{\phi(t)\}^2}}$$

$$= -\mathrm{Lt}\frac{f'(t)}{\phi'(t)} \cdot \frac{[\phi(t)]^2}{f(t)}.$$

In the limit, $\dfrac{\phi(t)}{f(t)}$ takes the form $\dfrac{0}{0}$; and the value of it is the same as that of $\dfrac{\phi'(t)}{f'(t)}$. If this value is finite, say $= a$, then

$$\mathrm{Lt}\frac{f'(t)}{\phi'(t)} \cdot \frac{\phi(t)}{f(t)} \cdot \phi(t) = \mathrm{Lt}\frac{a}{a} \cdot \phi(t) = 0.$$

Hence, in the limit, $\log z = 0$, and $z = 1$.

If, however, $\dfrac{\phi(0)}{f(0)}$ is either infinitely small or infinitely large, the above expression takes the forms $\dfrac{0}{0}$ and $\dfrac{\infty}{\infty} \times 0$, respectively; and we are unable without further investigation to draw any conclusion as to the limiting value of $z$.

As a particular case, suppose that $\phi(t) = t^2$, $f(t) = t$, so that $\dfrac{\phi(t)}{f(t)} = t$, and $\dfrac{\phi(0)}{f(0)} = 0$; then

$$\mathrm{Lt}\frac{f'(t)}{\phi'(t)} \cdot \frac{[\phi(t)]^2}{f(t)} = \mathrm{Lt}\frac{1}{2t} \cdot \frac{t^4}{t} = \mathrm{Lt}\frac{t^2}{2} = 0;$$

so that in this case also, $\mathrm{Lt}z = 1$.

Next suppose $\phi(t) = t, f(t) = t^2$; so that $\dfrac{\phi(t)}{f(t)} = \dfrac{1}{t}$, and $\dfrac{\phi(0)}{f(0)} = \infty$; then $\mathrm{Lt}\log z = \mathrm{Lt}\dfrac{2t}{1} \cdot \dfrac{t^2}{t^2} = \mathrm{Lt}2t = 0$; and again $\mathrm{Lt}z = 1$.

In every case we have as yet considered, the limit of $z$ is 1; and we can prove that it has the same value whenever $f(t)$ and $\phi(t)$ are ordinary algebraic functions; that is to say, when neither of them is a transcendental function. In this case, since $f(0) = 0$ and $\phi(0) = 0$, $f(t)$ and $\phi(t)$ must both be capable of expansion in series containing only positive powers of $t$, so that

$$f(t) = at^m + bt^n + \ldots\ldots$$

$$\phi(t) = \alpha t^\mu + \beta t^\nu + \ldots\ldots$$

where the indices are all positive, and $t^m$, $t^\mu$, are the lowest powers of $t$ in the two series respectively, then

$$f'(t) = mat^{m-1} + nbt^{n-1} + \ldots\ldots$$
$$\phi'(t) = \mu a t^{\mu-1} + \nu\beta t^{\mu-1} + \ldots\ldots$$

and $$\mathrm{Lt}\frac{f'(t)}{\phi'(t)}\cdot\frac{[\phi(t)]^2}{f(t)} = \mathrm{Lt}\frac{tf'(t)}{f(t)}\cdot\frac{\phi(t)}{t\phi'(t)}\cdot\phi(t) = \frac{m}{\mu}\times 0 = 0\,;$$

so that, as before, $\mathrm{Lt}z = 1$.

We can furthermore prove that, whenever $\dfrac{\phi(0)}{f(0)}$ is zero, the limit of $z$ is 1. For

$$\log z = \phi t\log(ft) = \frac{\phi t}{ft}\times ft\log(ft),$$

$$\text{and } \mathrm{Lt}\ \log z = \mathrm{Lt}\frac{\phi t}{ft}\times\mathrm{Lt}\,ft\log(ft).$$

$$\text{But } \mathrm{Lt}\,ft\log(ft) = \mathrm{Lt}\frac{\log(ft)}{\dfrac{1}{ft}} = \mathrm{Lt}\frac{\dfrac{f't}{ft}}{-\dfrac{f't}{(ft)^2}} = -\mathrm{Lt}\,ft$$
$$= 0.$$

Hence $\mathrm{Lt}\ \log z = 0\times 0 = 0$, and $\mathrm{Lt}\ z = 1$. The same proof may be employed in the case when $\dfrac{\phi(0)}{f(0)}$ is finite.

That we must not from these results conclude that the limit of $z$ is always 1, at once appears from the fact that, by giving a suitable form to $\phi(t)$, we may make the limit have any finite positive value, say $c$. In fact, suppose that $f(t)$, $\phi(t)$, are such functions that for *all* values of $t$ $\{f(t)\}^{\phi(t)}$ is constant and $= c$.

Then $\phi(t) = \dfrac{\log c}{\log f(t)}$; and, if $f(0) = 0$, $\phi(0) = \dfrac{\log c}{-\infty} = 0$;

or $f(t)$, $\phi(t)$, both vanish with $t$. In this case we have for all values of $t$, $z = (ft)^{\phi t} = c$; and the same is still true in the limit when $t = 0$.

The value of $\dfrac{\phi(0)}{f(0)}$ is now infinite; for

$$\mathrm{Lt}\frac{\phi(t)}{f(t)} = \mathrm{Lt}\frac{\log c}{f(t)\log f(t)} = \log c\,\mathrm{Lt}\frac{\dfrac{1}{f(t)}}{\log f(t)}$$
$$= \log c\,\mathrm{Lt}\frac{-\dfrac{f'(t)}{[f(t)]^2}}{\dfrac{f'(t)}{f(t)}} = -\log c\,\mathrm{Lt}\,\frac{1}{f(t)}$$
$$= \pm\infty\,;$$

the upper or lower sign being used according as $\log c$ is negative or positive.

It is easy to assign other forms to $\phi(t)$ that will make the limit of $z$ different from 1. Taking the expression

$$\phi(t)=\frac{\log c}{\log f(t)},$$

if we add to both numerator and denominator quantities which ultimately vanish in comparison with them, the limit of $z$ will still be $c$. Thus, let us put ($n$ being positive and not $<1$),

$$\phi(t)=\frac{\log c}{t^n+\log f(t)};$$

then $$\frac{1}{\phi(t)}=\left\{t^n+\log f(t)\right\}\div\log c;$$

$\therefore$ $$-\frac{\phi'(t)}{[\phi(t)]^2}=\left\{nt^{n-1}+\frac{f'(t)}{f(t)}\right\}\div\log c;$$

and $$-\frac{f'(t)}{f(t)}\cdot\frac{[\phi(t)]^2}{\phi'(t)}=\frac{f'(t)}{f(t)}\cdot\frac{\log c}{nt^{n-1}+\dfrac{f'(t)}{f(t)}}=\frac{\log c}{nt^{n-1}\dfrac{f(t)}{f'(t)}+1}.$$

Now, if $f(0)=0$, then $\frac{f(0)}{f'(0)}=0$; for, $t$ being the abscissa ON, (Fig. 16, 17) and $f(t)$ the ordinate PN, of a curve which passes through the origin, $\frac{f(t)}{f'(t)}$ is the subtangent TN, which evidently vanishes when P moves up to the origin, that is, when $t=0$.

Hence $-\mathrm{Lt}\frac{f'(t)}{f(t)}\cdot\frac{[\phi(t)]^2}{\phi'(t)}=\log c$; or, in the limit, $\log z=\log c$, and $z=c$.

It will be found that, as in the last case, $\frac{\phi(0)}{f(0)}=-\infty$, if $c>1$; and $=+\infty$, if $c<1$.

If we next desire to give $\phi(t)$ such a form that the limit of $z$ will be zero, we must examine what takes place when we suppose $c$ to become indefinitely small. In this case $\log c$ becomes indefinitely large, and negative; but since $\phi(t)$, which $=\frac{\log c}{\log f(t)}$, is to vanish with $t$, $\log c$ must vanish in comparison with $\log f(t)$. We must therefore substitute for $\log c$ a function which, when $t=0$, becomes $-\infty$, but vanishes in comparison with $\log f(0)$. Such a function is $-\log\{-\log f(t)\}$. We will put, then, $n$ being positive,

$$\phi(t)=\frac{-\log\{-\log f(t)\}}{mt^n+\log f(t)}.$$

Hence $$\log z = \phi(t)\log f(t) = \frac{-\log\{-\log f(t)\}}{mt^n + \log f(t)} \,.\, \log f(t)$$
$$= \frac{-\log\{-\log f(t)\}}{\dfrac{mt^n}{\log f(t)} + 1};$$
whence Lt $\log z = -\infty$, and Lt $z = 0$. In this case
$$\text{Lt}\frac{\phi(t)}{f(t)} = -\text{Lt}\frac{\log\{-\log f(t)\}}{f(t)[mt^n + \log f(t)]} = \frac{-\infty}{-0} = +\infty;$$
since $mt^n$ vanishes in comparison with $\log f(t)$, and Lt $f(t).\log f(t) = -0$.

Similarly, if we put
$$\phi(t) = \frac{+\log\{-\log f(t)\}}{mt^n + \log f(t)},$$
we shall get Lt$z = \infty$, and Lt $\dfrac{\phi(t)}{f(t)} = -\infty$.

We have thus proved that when $\dfrac{\phi(0)}{f(0)}$ is either zero or finite, whether positive or negative, the limit of $z$ (or $(f0)^{\phi 0}$) is $= 1$; and when $\dfrac{\phi(0)}{f(0)} = +\infty$, the limit may have any value from 0 to 1; and when $\dfrac{\phi(0)}{f(0)} = -\infty$, any value from 1 to $\infty$.

It will help us to understand these results better if we study the form of the surface, $z = x^y$. Giving to $x$ any finite positive value, $+a$, and to $y$ any finite value, $b$, which may be either positive or negative, we get one real and positive value of $z$, namely $a^b$; and the question we have been considering is the determination of the value of $z$ when both $a$ and $b$ simultaneously diminish according to a given law, and ultimately vanish together; or in geometrical language, the determination of the point or points, where the $z$-axis cuts the surface. We consider only positive values of $x$, because functions of the form $(-a)^b$ will not give us a continuous series of real values as $b$ varies continuously; and we neglect all the negative values of $z$, that arise from giving to $b$ such values as $\frac{1}{2}$, $\frac{3}{4}$, &c.; or, in general, $\dfrac{2m+1}{2n}$; for these values also do not form a continuous series. Reverting to our old supposition that $x = f(t)$, $y = \phi(t)$, $f(0) = 0$, and $\phi(0) = 0$, then any value of $t$ determines a point in the $xy$ plane; and the two equations together determine a curve in that plane passing through the origin. This curve may be considered as the projection of a curve on the surface; and if we suppose $t$ to be diminished until it vanishes, and then determine the limiting value of $z$, this is equivalent to determining what point on the $z$-axis we shall arrive at, if we travel to it along an assigned line on the surface. When we proved above that, if $\frac{\phi(0)}{f(0)}$ is finite or zero, then $z_0 = 1$, we virtually prove that, if we travel along any line on the surface, the projection of which on the $xy$ plane passes through the origin and does not touch the $y$-axis, we shall always arrive at the point on the $z$-axis at a distance $= 1$ from the origin. When we next proved that, if $f(t)$, $\phi(t)$, are algebraic functions, the value of $z_0$ is still $= 1$, although $\frac{\phi(0)}{f(0)}$ is infinite, we virtually proved that, if we travel along any curve on the surface, the projection of which is any algebraic curve passing through the origin and touching the $y$-axis, we still arrive at the same point. We have also proved that, by proceeding along a suitable curve on the surface, we may arrive at any point we please on the $z$-axis; but in this case, the projection of the curve on the $xy$ plane, must have contact of an infinitely high order with the $y$-axis.

In order further to elucidate the subject, I will now consider the nature of the sections of the surface made by the plane, $z = c$, for different positive values of $c$. The equation to the projection of this section on the $xy$ plane, is $x^y = c$; whence $y = \frac{\log c}{\log x}$. When $c < 1$, the curve represented by this equation is of the form shown in Fig. 18, where the positive part of the $y$-axis is drawn downwards because this is more convenient than the usual position, when we desire to picture to ourselves the various sections as arranged on the surface, and thus obtain a conception of the configuration of the latter. The curve passes through the origin and touches the positive part of the $y$-axis; and the line $x = 1$ is an asymptote. If we put $y = 1$, we get $x = c$; and if $y = -1$, we get $x = \frac{1}{c}$. Hence, if we make in the figure, $\mathrm{OB} = \mathrm{O}b = \mathrm{OA} = 1$, and draw BP, $bp$, parallel to the $x$-axis and meeting the curve in P, $p$, and then draw the perpendiculars PD, $pd$; we have $\mathrm{OD} = c$, and $\mathrm{O}d = \frac{1}{c}$.

When $c > 1$, the curve has the same form but it lies on the other side of the $x$-axis; and if the new value, of $c$ (say $c_1$) is the reciprocal of the old one, the new curve will be the reflection of the old one on the $x$-axis.

We have seen that, whatever positive value we give to $c$, other than unity, the curve in Fig. 18 passes through the origin; and we conclude that every point on the positive part of the $z$-axis, other than the point for which $z = 1$, lies on the surface. The excepted point for which $z = 1$ requires special consideration; in this case, we have $x^y = 1$, and we see that when $x = 1$ this is satisfied by any value of $y$; hence every point on the line determined by $x = 1$, $z = 1$ lies on the surface. The equation is also satisfied by putting for $x$ any finite positive quantity and making $y = 0$; in other words, every point on the positive part of the line, $y = 0$, $z = 1$, lies on the surface; but we have still to consider the point (001).

The best way of examining the form of the surface in the neighbourhood of this point, is to study the changes in the section of the surface by the plane $z = c$, as $c$ gradually approaches unity. For this purpose, we may neglect the part of the curve to the right of the asymptote. It is easily found that the coordinates of the point of inflection are $x = \epsilon^{-2} = \cdot 1353$, and $y = -\frac{1}{4} \log c$; and that the equation to the tangent at that point is

$$\frac{x}{e^{-2}} + \frac{4y}{\log c} + 1 = 0.$$

Hence, if we take OE = OF = ·1353 (Fig. 19), OA as before being unity, and draw EH parallel to O$y$, the points of inflection of the various sections will all lie in EH, and the tangents at them will all pass through the point F. We are now able to see that, as $c$ increases up to 1, the curve of section gradually approximates to the straight lines OA and AG; but, however close the curve is to OA, so long as it does not actually coincide with OA, it touches O$y$. It is also easy to see that, when $c > 1$, the curves lie on the negative side of the $x$-axis, and all touch the negative part of the $y$-axis.

These geometrical results show that when $x$ and $y$ both vanish, if the ultimate value of the ratio $y : x$ is either zero or finite, whether positive or negative, then the ultimate value of $x^y$ is unity. Let PO be the projection on the $xy$ plane, of the path of the moving point on the surface, and suppose that the surface is cut by an indefinitely large number of planes parallel to the $xy$ plane, then, as $t$ receives different values, the path of the moving point on the surface will generally cut these curves of section; and the same will be true of the projections of the path and of the sections on the $xy$ plane; and the last of these curves which the path meets, will determine the ultimate value of $z$. Now, since all the projections of the sections touch the $y$-axis at the origin, if OP, the projection of the path of the moving particle, does not touch the $y$-axis at the origin, it must, as it approaches the origin, cut the projections of all the sections in the neighbourhood of the origin, until it meets at the origin, the line OA, which is the projection of the section for which $z=1$. Hence, if OP does not touch the $y$-axis at the origin, or if the ultimate value of $y:x$ {or of $\phi(t):f(t)$} is not $\pm\infty$, then the ultimate value of $z$ is unity. This quite agrees with what was proved in the earlier part of this paper.

Dr Thomas Muir has furnisht me with the following references to papers which I have not myself seen:—

Johnson, W. W. On the expression $0^0$. *Analyst* III, pp. 118-121 (1876).

Franklin, F. Note on indeterminate exponential forms. *American Journal of Mathematics* I, pp. 368-9 (1878).

## The so-called Simson line.

By John Alison, M.A.

This paper is an attempt to collect and arrange some of the propositions regarding the so-called Simson line, contained in various Mathematical Treatises and Journals. Proofs have been altered, or new ones substituted to suit the arrangement.

§ 1. *Figure* 20. If from any point on the circumference of a circle perpendiculars be drawn to the sides of an inscribed triangle, the feet of these perpendiculars are collinear. [The line on which they lie is known as the Simson line; but it is doubtful if Simson knew it. See *Nature*, Vol. XXX. p. 635.]

§ 2. There are two theorems of which the preceding may be regarded as a particular case.

(*a*). If lines be drawn from any point on the circumference of a circle, making equal angles in the same direction with the three sides of an inscribed triangle, the points of intersection of these lines with the respective sides are collinear. Poncelet, *Propriétés Projectives*, § 468.

(*b*). *Figure* 21. If from any point P in a plane perpendiculars be drawn to the three sides of a triangle ABC; and if the feet of these perpendiculars be joined, the locus of P so that the area of the pedal triangle may be constant is a circle concentric with the circle ABC.

As P moves from the centre of the circle ABC to its circumference, the area of the pedal triangle diminishes from $\frac{1}{4}$ (triangle ABC) to zero when the triangle becomes the Simson line.

Let QRS, the pedal triangle, be denoted by $\Delta'$, and the triangle formed by joining the middle points of the sides of the triangle ABC by $\Delta$; the radius of the circle ABC, by R, and the distance between P and the centre of the circle ABC, by R′. Let O be the centre of the circumscribed circle, and AOK a diameter. Join CP and produce it to meet the circumference of the circle in D and draw BD. Join OP and CK, and draw the perpendiculars AH, BI, PE, RF to the lines BC, CA, BD, QS respectively, and produce QP to meet BA in G.

Then since each of the quadrilaterals BSPQ, PQCR is inscriptible in a circle, we have $\angle DBA = \angle DCA = \angle PQR$, and $\angle ABP = \angle PQS$; hence $\angle EBP = \angle FQR$, and the triangles EBP, FQR are similar; therefore $\frac{PE}{RF} = \frac{PB}{RQ}$ or $PE \cdot RQ = PB \cdot RF$.

From the similar triangles BPG and GSQ we get

$$\frac{QS}{BP} = \frac{GS}{GP} = \frac{GQ}{GB} = \frac{AH}{AB} = \frac{AC}{AK}; \text{ therefore } \frac{QS}{BP} = \frac{AC}{AK} \text{ or } \frac{QS \cdot RF}{BP \cdot RF} = \frac{AC}{AK}.$$

$$\text{Similarly } \frac{RQ}{CP} = \frac{AB}{AK} \quad \dots \quad \dots \quad \dots \quad \dots \quad (1).$$

Again, the angles EDP, BAC being equal, we have

$$\frac{PE}{PD} = \frac{BI}{AB} \quad \dots \quad \dots \quad \dots \quad \dots \quad (2).$$

$$\text{By (1) and (2) } \frac{RQ \cdot PE}{CP \cdot PD} = \frac{AB \cdot BI}{AB \cdot AK} = \frac{BI}{AK}.$$

But it has been shown that $RQ \cdot PE = PB \cdot RF$; $\therefore \frac{PB \cdot RF}{CP \cdot PD} = \frac{BI}{AK}$;

and since $\frac{QS \cdot RF}{PB \cdot RF} = \frac{AC}{AK}$, therefore $\frac{QS \cdot RF}{CP \cdot PD} = \frac{AC \cdot BI}{AK^2}$.

Now $QS \cdot RF = 2\Delta'$, $CP \cdot PD = DO^2 - OP^2 = R^2 - R'^2$, $AC \cdot BI = 8\Delta$, and $AK^2 = 4R^2$; consequently we get finally $\frac{\Delta'}{\Delta} = \frac{R^2 - R'^2}{R^2}$.

In a similar manner when the point P is without the circumscribed circle it may be proved that $\frac{\Delta'}{\Delta} = \frac{R'^2 - R^2}{R^2}$; consequently $\frac{\Delta'}{\Delta} = \pm \frac{R^2 - R'^2}{R^2}$ according as the point is *within* or *without* the circumscribed circle. When P is on the circumscribed circle $R = R'$; hence $\Delta' = 0$, that is, the triangle QRS becomes a straight line.

*The Mathematician*, (1847) Vol. II. p. 37 (Davis). An analytical solution is given in Vol. I. (1843).

§ 3. If the projections (orthogonal or oblique) of a point on the three sides of a triangle are collinear, the locus of the point is the circumscribed circle of the triangle.

§ 4. If from any point on a circle three chords be drawn, and if on the same side of these chords three similar segments of circles be described, the three circles so constructed will intersect two by two on a straight line.

Conversely, if three circles have one common point, and intersect again two by two on a straight line; and if chords be drawn from the common point cutting off similar segments from the three circles in the same direction, the extremities of these three chords shall be concyclic with the common point.

§ 5. *Figure* 20. If three circles pass through one point which is concyclic with their centres, they shall intersect two by two on a straight line.

Let P be the common point, X, Y, Z the centres, draw diameters through P to A, B, C respectively.

P is the external centre of similitude of the circles round XYZ and ABC; and it is on the circle XYZ; therefore it is on the circle ABC The theorem then becomes a case of § 4.

The points of intersection lie on the Simson line P(ABC).

Cor. The Simson line P(ABC) makes, with the sides of the triangle ABC taken by twos, three triangles, the centres of whose circumscribed circles are concyclic with P on a circle of half the radius of the circle ABC. *Nouvelles Annales de Mathématiques*, 2nd Series, Vol. X., p. 206.

§ 6. *Figure* 22. For every complete quadrilateral there can be found a point such that the feet of the perpendiculars drawn from it on the four sides are collinear.

For the locus of the point, the feet of whose perpendiculars on AB, BC, CD are collinear, is the circle round CBE (§ 3); and the locus of the point, the feet of whose perpendiculars on BC, CD, DA are collinear, is the circle round CDF.

Therefore the feet of the perpendiculars from P, the second point of intersection of the two circles on the four lines AB, BC, CD, DA are collinear.

Cor. 1. The circumscribed circles of the four component triangles of any complete quadrilateral pass through one point.

Cor. 2. For every complete quadrilateral there is a point P such that if it be joined to the six vertices, and if circles be described on these lines as diameters, the six circles intersect three by three in four points which are collinear.

The line of collinearity is the Simson line of each of the component triangles with respect to P, and may be called the Simson line of the complete quadrilateral. Catalan's *Théorèmes et Problèmes*, 6th ed., p. 34.

§ 7. *Figure* 20. If a triangle ABC inscribed in a circle have its vertex A fixed; and if the sides AB, AC be such that the rectangle under their distances from a fixed point P on the circle is constant; the Simson line P(ABC) meets BC on a fixed circle.

Draw a tangent at A, and on it drop a perpendicular PD′.

P,B′,A,C′,D′ are concyclic, and therefore $\angle PD'B' = \angle PC'B'$, and $\angle D'B'P = \angle D'AP = \angle ACP$, the angle in the alternate segment, $= \angle B'A'P$, the angle in the same segment of the circle PB′A′C.

Since $\angle PD'B' = \angle PC'A'$ and $\angle PB'D' = PA'C'$, therefore the triangles PB′D′ and PC′A′ are similar and $PD'\cdot PA' = PB'\cdot PC'$.

Since A and P are fixed points on the circle, therefore PD′ is constant; and if $PB'\cdot PC'$ be constant PA′ is constant; that is, A′ lies on a circle whose centre is P. PA′ the radius $= \dfrac{PB'\cdot PC'}{PD'}$. Since PA′B is a right angle BC has this circle for its envelope.

Cor. If DEF be the triangle whose sides touch the circle at A, B, and C, and if PD′, PE′, PF′ be the perpendiculars from P on EF, FD, and DE respectively; $PA'\cdot PB'\cdot PC' = PD'\cdot PE'\cdot PF'$.

For $PD'\cdot PA' = PB'\cdot PC'$,
$PE'\cdot PB' = PC'\cdot PA'$,
and $PF'\cdot PC' = PA'\cdot PB'$.
Hence $PD'\cdot PE'\cdot PF' = PA'\cdot PB'\cdot PC'$.

§ 8. *Figure* 23. The Simson line D(ABC) is parallel to AP, BQ, and CR; P, Q, R being the points where the perpendiculars from D on BC, CA, AB meet the circle again.

$$\begin{aligned}\angle BAP = \angle BDP &= \text{complement of } \angle DBC,\\ &= \text{complement of } \angle DAB',\\ &= \text{complement of } \angle DC'B',\\ &= \angle AC'B'.\end{aligned}$$

Therefore AP and C′A′ are parallel; hence also BQ, CR are parallel to D(ABC). Catalan's *Théorèmes et Problèmes*, p. 36.

Name the othocentres of the four triangles ABC, BCD, CDA, DAB, $O_4, O_1, O_2, O_3$ respectively.

§ 9. *Figure* 23. The Simson line D(ABC) bisects $DO_4$.

Since AP is parallel to C′A′, therefore MAPA′ is a parallelogram, and $AM = A'P = A'O_1$.

$O_4$, $O_1$ are the images of N and P in BC; therefore $O_4O_1$ and NP are antiparallels with respect to BC. But AD is also antiparallel to NP with respect to BC; therefore $O_4O_1DA$ is a parallelogram, and $AO_4 = DO_1$; $AM + AO_4 = DO_1 + O_1A'$; $MO_4 = DA'$; therefore $MO_4A'D$ is a parallelogram, and $DO_4$ one of the diagonals is bisected by the other diagonal D(ABC). Catalan's *Théorèmes et Problèmes*, p. 37.

§ 10. *Figure* 22. The orthocentres of the four component triangles of a complete quadrilateral are collinear.

For if P the intersection of the four circumscribed circles be joined to the four orthocentres $O_1$, $O_2$, $O_3$, $O_4$, the lines $PO_1$, $PO_2$, $PO_3$, $PO_4$ are bisected by the Simson lines of the triangles with respect to P (§ 9). But these Simson lines are coincident (§ 6, Cor. 2); the points of bisection of $PO_1$, $PO_2$, $PO_3$, $PO_4$ are collinear; therefore $O_1$, $O_2$, $O_3$, $O_4$ lie on a line parallel to the Simson line of the complete quadrilateral.

§ 11. *Figure* 22. The Simson line of a complete quadrilateral is at right angles to the line passing through the middle points of the three diagonals.

$O_1$ is the orthocentre of ABF, X, Y, Z the points where $O_1A$, $O_1B$, $O_1F$ cut the sides opposite A, B, F at right angles. AX, BY, FZ are pairs of points on the circles on AC, BD, FE as diameters; therefore the potencies of the three circles with respect to $O_1$ are $O_1A \cdot O_1X$, $O_1B \cdot OY$, $OF \cdot OZ$. But since AXBY and BYFZ are sets of concyclic points; $O_1A \cdot O_1X = O_1B \cdot O_1Y = O_1F \cdot O_1Z$; that is, $O_1$ is a point on the radical axis of the three circles on the diagonals as diameters.

Hence also $O_2$, $O_3$, $O_4$ are on the radical axis of the three circles, which is at right angles to the line through their centres, the middle points of the three diagonals. The Simson line is parallel to $O_1O_2O_3O_4$ (§ 10) and is therefore at right angles to the line joining the middle points of the diagonals.

§ 12. *Figure* 23. Given four points A, B, C, D on a circle, the Simson lines A(BCD), B(CDA), C(DAB), D(ABC) are concurrent.

$AO_4O_1D$ is a parallelogram (§ 9) and T is the intersection of the diagonals; therefore T bisects $AO_1$ and $DO_4$. Hence also T bisects $BO_2$ and $CO_3$. But the lines $AO_1$, $BO_2$, $CO_3$, $DO_4$ are bisected by the Simson lines A(BCD), B(CDA), C(DAB), D(ABC) respectively; therefore the four Simson lines pass through T. *Educational Times Reprint*, Vol. I., Question 1431.

Cor. I. Since $O_1$, $O_2$, $O_3$, $O_4$ are the external centres of similitude of the circumscribed circle ABC and the nine-point circles of the triangles BCD, CDA, DAB, and ABC respectively, and the radius of the nine-point circle is half the radius of the circumscribed circle; hence the nine-point circles bisect $AO_1$, $BO_2$, $CO_3$, $DO_4$ respectively and so pass through T.

The nine-point circles of the four triangles formed by joining any four points in a plane are concurrent. See *Lady's and Gentleman's Diary*, 1864, p. 55.

Cor. 2. If A be a variable point on the circle, then the Simson lines D(ABC), B(ADC) intersect on the nine-point circle of DBC at a constant angle.

However A move the Simson lines of the four triangles are concurrent with their nine-point circles. But the triangle DBC is fixed; therefore its nine-point circle is the locus of the intersection of D(ABC) and B(ADC).

Since D(ABC), B(ADC) pass through the fixed points A′ and H respectively on the nine-point circle of DBC and intersect on the same circle, therefore their angle of intersection is constant.

The angle is equal to $\angle A'UH = 2 \angle A'DC$,

= twice the complement of $\angle BCD$.

§ 13. *Figure* 23. (*a*) $O_1O_2O_3O_4$ is a quadrilateral equal in all respects to ABCD.

For $O_4O_1$ has been shown to be parallel and equal to DA (§ 9); hence also $O_1O_2$ is equal and parallel to AB,

$O_2O_3$ „ „ BC,

and $O_3O_4$ „ „ CD.

Catalan's *Théorèmes et Problèmes.*

(*b*) T is the internal centre of similitude of the two quadrilaterals.

(*c*) If three points be taken from one of the sets of concyclic points (ABCD), ($O_1O_2O_3O_4$) and the non-corresponding point from the other set, and if any three of these four points be taken as vertices of a triangle, the fourth point is its orthocentre. For example, take the four points $ABCO_4$; $O_4$ is the orthocentre of ABC by construction, and by the same construction A is the orthocentre of $BCO_4$,

B „ $CO_4A$,

C „ $O_4AB$.

If three be chosen from the second set and one from the first the same is true, for ABCD and $O_1O_2O_3O_4$ are reciprocal figures; that is, the same construction which derived $O_1O_2O_3O_4$ from ABCD will derive ABCD from $O_1O_2O_3O_4$.

(*d*) Any pair of points from one of the sets (ABCD), ($O_1O_2O_3O_4$) with the non-corresponding pair from the other set, are four points on a circle whose radius is equal to that of ABCD or $O_1O_2O_3O_4$. For example, $O_1BCO_4$ are on an arc which is the image of BNC in BC, and so is part of a circle of equal radius with ABC.

§ 14. *Figure* 23. The Simson line D(ABC) is identical with the Simson line $O_1(BCO_4)$.

For D(ABC) passes through A′ the projection of D on BC, and through T the middle point of $DO_4$ (§ 9); and $O_1(BCO_4)$ passes through A′ the projection of $O_1$ on BC, and through T the middle point of $O_1A$.

Hence also D(ABC) coincides with $O_2(CO_4A)$ and $O_3(O_4AB)$. *Journal de Mathématiques Spéciales*, 2nd Series, Vol. III. p. 30 (Weill).

Cor. 1. Every line in the quadrilateral ABCD is parallel to the corresponding line of $O_1O_2O_3O_4$, and if any line of one pass through T, the centre of similitude, so does the corresponding line of the other.

Hence D(ABC) and $O_4(O_1O_2O_3)$ are parallel to each other, and both pass through T; therefore they coincide.

This gives a set of eight Simson lines which are coincident, viz.:—

(1) D(ABC), $O_1(BCO_4)$, $O_2(CO_4A)$, $O_3(O_4AB)$,
$O_4(O_1O_2O_3)$ $A(O_2O_3D)$, $B(O_3DO_1)$, $C(DO_1O_2)$.

Of the points ABCD, $O_1O_2O_3O_4$ we may form eight sets of four concyclic points, and of the four points in a set we may form four triangles and construct the Simson line of each triangle with respect to the fourth concyclic point. Thus we get thirty two Simson lines which coincide as shown above in four sets of eight, all passing through T.

The other sets of coincident Simson lines are

(2) A(BCD), $O_2(CDO_1)$, $O_3(DO_1B)$, $O_4(O_1BC)$,
$O_1(O_2O_3O_4)$, $B(O_3O_4A)$, $C(O_4AO_2)$, $D(AO_2O_3)$.

(3) B(CDA), $O_3(DAO_2)$, $O_4(AO_2C)$, $O_1(O_2CD)$.
$O_2(O_3O_4O_1)$ $C(O_4O_1B)$, $D(O_1BO_3)$, $A(BO_3O_4)$.

(4) C(DAB), $O_4(ABO_3)$, $O_1(BO_3D)$, $O_2(O_3DA)$,
$O_3(O_4O_1O_2)$ $D(O_1O_2C)$, $A(O_2CO_4)$, $B(CO_4O_1)$.

Cor. 2. The nine-point circles of the thirty-two triangles named all pass through T; and they coincide in eight sets of four circles; for the nine-point circle of ABC is also the nine-point circle of

$BCO_4$, $CO_4A$, and $O_4AB$.

§ 15. *Figure* 23. Any line through T bisecting one of the sides of the quadrilateral ABCD is at right angles to the opposite side; and any line through T bisecting one diagonal is perpendicular to the other diagonal.

Since $AO_4O_1D$ is a parallelogram, T the intersection of diagonals and X the middle point of one side; therefore XT is parallel to $AO_4$ and so perpendicular to BC.

Y is the middle point of DB, and T the middle point of $DO_4$; therefore YT is parallel to $BO_4$ and so perpendicular to AC.

Cor. If S be the centre of the circumscribed circle and W the middle point of BC; XW and ST bisect each other.

For XT and SW are both perpendicular to BC; and SX and TW are both perpendicular to AD. Therefore SWTX is a parallelogram, and ST, WX, the diagonals bisect each other. *Lady's and Gentleman's Diary* for 1865, p. 60.

§ 16. *Figure* 23. The point T is such that

$$TA^2+TB^2+TC^2+TD^2=[\text{Diameter of } ABC]^2.$$

Draw CJ at right angles to BC. BJ is a diameter. $CJ=AO_4$.

Because T is the middle point of $DO_4$; therefore

$$\begin{aligned} 2(TA^2+TD^2) &= DA^2+AO_4^2=DA^2+CJ^2\\ &= DA^2+BJ^2-BC^2,\end{aligned}$$

also $$2(TB^2+TA^2)=AB^2+BJ^2-CD^2,$$

$$2(TC^2+TB^2)=BC^2+BJ^2-DA^2,$$

and $$2(TD^2+TC^2)=CD^2+BJ^2-AB^2.$$

Hence $$TA^2+TB^2+TC^2+TD^2=BJ^2.$$

*Educational Times Reprint*, Vol. I. Question 1431.

§ 17. *Figure* 24. The angle between two Simson lines D′(ABC) and D″(ABC) is equal to half of $\angle D'SD''$ where S is the centre of the circumscribed circle of ABC.

$$\begin{aligned} \angle T &= \angle TA'B-\angle TA''B,\\ &= \angle B'D'C-\angle B''D''C, \text{(since } A'B'CD', A''B''CD''\end{aligned}$$

are inscribable quadrilaterals),

$$\begin{aligned} &= \angle D''CA-\angle D'CA,\\ &= \angle D''CD'=\frac{\angle D''SD'}{2}.\end{aligned}$$

Cor. 1. If BD′ and CD″ meet in M outside the circle; $\angle T=\angle BAC-\angle M$. If M be inside the circle, $\angle T=\angle M-\angle A$.

If M any point in the plane be joined to A,B,C, cutting the circle in D′, D″, D‴, and if the three Simson lines relative to the points D′, D″, D‴ be drawn; then,

$$\begin{aligned} \angle T &= \pm(\angle A-\angle M),\\ \angle T' &= \pm(\angle B-\angle M'),\\ \angle T'' &= \pm(\angle C-\angle M'');\end{aligned}$$

$$\therefore\quad \angle T+\angle T'+\angle T''=\pm(\text{two right angles}-\overline{\angle M+\angle M'+\angle M''});$$

$$\therefore\quad \angle M+\angle M'+\angle M''\pm(\angle T+\angle T'+\angle T'')=\text{two right angles}.$$

*Educational Times Reprint*, Vol. IX. Question 2480 (Tucker).

Cor. 2. If E, F, G be the vertices of any triangle inscribed in the circle ABC, the Simson lines E(ABC), F(ABC), G(ABC) make a triangle similar to EFG.

Cor. 3. This theorem gives us a method of finding a point on the circumscribing circle whose Simson line with respect to the given triangle will be parallel to a given line.

§ 18. *Figure* 24. The Simson lines relative to two points diametrically opposite intersect at right angles on the nine-point circle.

If D′ and D″ be diametrically opposite then $\angle$ D′SD″ = a straight angle; and the angle between D′(ABC) and D″(ABC) = $\dfrac{\angle \text{D}'\text{SD}''}{2}$ (§ 17) = one right angle.

The Simson line D′(ABC) passes through P′ the middle point of D′O; D″(ABC) passes through P″.

Since O is a centre of similitude of the two circles, when D′D″ becomes a diameter of ABC, P′P″ becomes a diameter of the nine-point circle; and the two lines passing through P′ and P″ and containing a right angle must intersect on the nine-point circle.

Cor. Two Simson lines at right angles intercept on any side of the triangle ABC between their points of intersection with the side and the extremities of the side two equal segments. For the points of intersection are the projections of the ends of a diameter. *Journal de Mathématiques Spéciales*, 2nd Series, Vol. III. p. 16.

§ 19. If ABC, A′B′C′ be two triangles in a circle having the pairs of vertices AA′, BB′, CC′ diametrically opposite, the two Simson lines P(ABC), P(A′B′C′) are at right angles to each other.

Since the triangle A′B′C′ is derived from ABC by rotating it through two right angles, therefore AH and A′H′ or A(ABC), A′(A′B′C′) are parallel.

P(ABC) makes with A(ABC) the angle $\dfrac{\text{PSA}}{2}$. (§ 17)

P(A′B′C′) makes with A′(A′B′C′) the angle $\dfrac{\text{PSA}'}{2}$.

The angle between P(ABC) and P(A′B′C′) is therefore

$$\frac{\angle \text{PSA}' + \angle \text{PSA}}{2} = \frac{\angle \text{ASA}'}{2} = \text{a right angle.}$$

Cor. If P′ be the diametral point of P, then the four Simson lines P(ABC), P′(ABC), P(A′B′C′), P′(A′B′C′) form a rectangle, two of whose vertices are on the nine-point circles of ABC and A′B′C′. *Educational Times Reprint*, Vol. VI. Question 1939. (Tucker).

§ 20. *Figure* 24. Two Simson lines D′(ABC) and D″(ABC) intercept on the nine-point circle of the triangle ABC two arcs, one of which is double of the other.

Since O the orthocentre of the triangle ABC is the external centre of similitude of the circumscribed and the nine-point circles, the arc P'P" is similar to the arc D'D", and subtends an angle at Q" equal to the angle in the segment D'BD".

But ∠T between the Simson lines is equal to the angle in the segment D'BD". Therefore ∠T is equal to ∠P'Q"P"; ∠Q'P'Q" is double of ∠P'Q"P"; and the arc Q'Q" is double of the arc P'P". *Journal de Mathématiques Spéciales*, 2nd Series, Vol. III. p. 13.

Cor. *Figure* 25. If the arcs of the nine-point circle be measured in the positive direction we have

$$\begin{aligned} XY &= 2VH = 2XW = 2WY = 2KU, \\ YZ &= 2WK = 2YU = 2UZ = 2LV, \\ ZX &= 2UL = 2ZV = 2VX = 2HW, \\ YX &= 2UV = 2YL = 2LX = 2HK, \\ ZY &= 2VW = 2ZH = 2HY = 2KL, \\ XZ &= 2WU = 2XK = 2KZ = 2LH, \\ & XH = KY + LZ. \end{aligned}$$

For the sides and perpendiculars of a triangle are the Simson lines of the three vertices and the three diametrically opposite points respectively.

§ 21. *Figure* 23. The length of each segment of the line D(ABC) made by the sides of the triangle ABC is proportional to the distance of D from the vertex of the angle opposite the segment.

For triangles ACD and C'DA' are similar;

therefore $\frac{DA'}{DC}=\frac{C'A'}{AC}$.

Triangles BDJ and DA'C are similar;

therefore $\frac{DA'}{DC}=\frac{BD}{BJ}$;

therefore $\frac{C'A'}{AC}=\frac{BD}{BJ}$; therefore $C'A'=\frac{AC}{2R}\cdot BD$.

Similarly $A'B'=\frac{BA}{2R}\cdot CD$.

$B'C'=\frac{CB}{2R}\cdot AD$.

[This theorem is a particular case of what is proved in § 2. (*b*), namely, $\frac{QS}{PB}=\frac{AC}{AK}$.]

Cor. 1. If D coincide with A then C′A′ becomes the perpendicular from A on BC and $p = \frac{AC\cdot AB}{2R}$. Hence $2R\cdot p = AB\cdot AC$, a well known theorem.

Cor. 2. Since $A'B' + B'C' + C'A' = 0$; hence taking account of sign we get Ptolemy's theorem that

$$AC\cdot BD = AB\cdot CD + BC\cdot AD.$$

Cor. 3. If two points D and D′ be taken symmetrical with respect to the diameter through B one of the vertices of the triangle ABC; the segments opposite B of the Simson lines with respect to the two points are equal.

For the segments are $\frac{AC}{2R}\cdot BD$ and $\frac{AC}{2R}\cdot BD'$, and $BD = BD'$.

Cor 4. If ABCD be a quadrilateral in a circle, the segments of
A(BCD) intercepted between BC and CD,
B(CDA) „ „ CD „ DA,
C(DAB) „ „ DA „ AB,
and D(ABC) „ „ AB „ BC

are equal, for they are each equal to $\frac{AC\cdot BD}{2R}$.

A(BCD) and C(DAB) are equally inclined to BD,
B(CDA) and D(ABC) „ „ „ „ AC.

§ 22. If D and D′ be symmetrical points with respect to the centre then the sum of the squares of the segments of their Simson lines relative to one of the vertices of the triangle ABC is equal to the square of the side opposite that vertex.

$$\overline{D(ABC)}^2 + \overline{D'(ABC)}^2 = \left(\frac{AC\cdot BD}{2R}\right)^2 + \left(\frac{AC\cdot BD'}{2R}\right)^2$$

$$= \left(\frac{AC}{2R}\right)^2 (BD^2 + BD'^2) = \left(\frac{AC}{2R}\right)^2 \cdot (2R)^2$$

$$= AC^2.$$

Cor. The sum of the squares of the six segments of the Simson lines with regard to two centrally symmetrical points is equal to $AB^2 + BC^2 + CA^2$. §§ 21, 22, *Journal de Mathématiques Elémentaires* Vol. II. pp. 68-71 (Julliard).

§ 23. *Figure* 23. The projection of any side of a triangle on a Simson line is equal to the segment of the line intercepted by the other two sides.

Draw BK perpendicular to AP. AK is equal to the projection of AB on D(ABC).

The triangles ABK and DEC are similar. For the arc PC + arc BAD = arc BA + arc LP; take away the common parts, and arc AD = arc CL; add arc DL to each, and arc AL = arc CD; hence $\angle$ ABK = $\angle$ DEC.

But the triangles are right angled, and are therefore similar.

Therefore $$\frac{AK}{AB}=\frac{CD}{DE}=\frac{CD}{2R}=\frac{A'B'}{AB}\ (\S\ 21);$$

and AK, the projection of AB on the Simson line, is equal to A′B′, the intercept between BC and CA.

§ 24. *Figure* 23. Since the perpendicular from B on D(ABC) cuts the circle in a point L so that LC = AD, we have the following theorem:—

If two points D, L be taken on the circle equidistant from A and C respectively, the line D(ABC) is perpendicular to LB and L(ABC) is perpendicular to DB.

Cor. When D is the middle point of AC, D(ABC) is at right angles to DB, the internal bisector of the angle ABC.

§ 25. *Figure* 26. ABCD is a square inscribed in a circle, and P a point on the circumference, the area of the quadrilateral formed by the Simson lines A(BCD), B(CDA), C(DAB), D(ABC) is equal to $\frac{1}{4}$(PA·PC + PB·PD).

ABC and ADC are diametral triangles, and therefore (§ 19) their Simson lines with respect to P cut one another at right angles. For the same reason the Simson lines of ABD and CBD intersect at right angles. By the construction of the figure the vertices of the quadrilateral lie on the sides and diagonals of the square.

$$(\S\ 21)\ A'B'=\frac{AB}{2R}\cdot PC,\ A'D'=\frac{CD}{2R}\cdot PA,$$

$$B'C'=\frac{CD}{2R}\cdot PB,\ C'D'=\frac{AB}{2R}\cdot PD.$$

But since the quadrilateral A′B′C′D′ is made up of two right-angled triangles, its area is equal to

$$\frac{A'B'\cdot A'D'+C'B'\cdot C'D'}{2}=\frac{AB^2}{8R^2}(PA\cdot PC+PB\cdot PD),$$

$$=\tfrac{1}{4}(PA\cdot PC+PB\cdot PD).$$

Cor. If ABCD be any rectangle in a circle, the area of the quadrilateral formed by the four Simson lines of P, if P be opposite AB or CD, is equal to

$$\tfrac{1}{4}(PA\cdot PC+PB\cdot PD).\frac{AD^2}{2R^2}.$$

*Educational Times*, Vol. XXXVI. p. 152. Question 7373. (Edwardes.)

§ 26. *Figure* 25. Let XYZ, HKL be the pedal and medial triangles respectively of the triangle ABC; U, V, W, the middle points of AO, BO, CO, where O is the orthocentre of ABC. (1) The Simson lines of the vertices of each of the triangles XYZ, HKL with respect to the other are concurrent at the centre of gravity of the perimeter of XYZ. (2) The Simson lines of the points U, V, W with respect to the triangle XYZ form a triangle, which has for its centre of similitude with ABC the centre of gravity of XYZ, and for its orthocentre the centre of gravity of the perimeter of XYZ.

(1) The Simson line K(XYZ) passes through the middle of XZ, and is parallel to BY, the internal bisector of XYZ. (§§ 18, 24, Cor.) But lines drawn through the middle points of the sides of a triangle parallel to the internal bisectors of the opposite angles are concurrent at the centre of gravity of the perimeter of the triangle.

Y(HKL) is the parallel drawn through D the middle of BY to the line BS, where S is the circumscribed centre of ABC, and orthocentre of HKL. (§ 9.) Let T be its point of intersection with PT drawn from the middle point of YZ parallel to AX. The triangles DTP and BYZ are respectively similar to ZOY and XYO; $\frac{ZY}{YO}=\frac{DP}{PT}=\frac{ZB}{2PT}$; $\frac{YZ}{YO}=\frac{ZB}{OX}$. Hence $XO=2PT$; therefore T is the centre of gravity of the perimeter of XYZ.

(2) U(XYZ), V(XYZ), W(XYZ) (§ 24, Cor.) are the parallels drawn through P, Q, R respectively to the sides BC, AC, AB; let M be the intersection of the two last. The line XP cuts QR in N; the line MN is parallel to AP and equal to $\frac{AP}{2}$. Then if MA cut NP in G, $NG=\frac{GP}{2}$; therefore G the centre of homology is the centre of gravity of XYZ.

T is the orthocentre of the triangle formed by these three Simson lines; because TM is parallel to AX.

*Mathesis*, Vol. V. p. 58 (Van Aubel); and *Educational Times*, Vol. 37, p. 380 (Tucker).

§ 27. *Figure* 25. The triangle formed by the three Simson lines U(XYZ), V(XYZ), W(XYZ) has its sides respectively parallel to and half of BC, CA, AB; so has the triangle UVW.

U, V, and W are the middle points of the arcs YZ, ZX, and XY. UX, VY, WZ are the internal bisectors of the angles of the triangle XYZ; and VW, WU, UV are at right angles to them respectively.

Hence the proposition—If XYZ be any triangle in a circle, U, V, W the middle points of the arcs cut off by the sides; the Simson lines U(XYZ), V(XYZ), W(XYZ) form a triangle whose sides are equal and parallel to those of the triangle UVW. *Educational Times*, Vol. XXXV. p. 289, Question 7186 (Tucker).

§ 28. *Figure* 23. Parabolas may be described such that A, B, C, D are their respective foci, and the sides of their corresponding triangles, tangents. The tangents at the vertices of these parabolas are concurrent, and their directrices pass through $O_1$, $O_2$, $O_3$, $O_4$ respectively.

For if D be the focus, and AB, BC, CA tangents, then the feet of the perpendiculars from D on the tangents lie on the tangent at the vertex; that is, the Simson line A′B′C′ is the tangent at the vertex of the parabola. The four Simson lines are the four tangents at the vertices, and they pass through T. (§ 12.)

Because $DO_4$ is drawn from the focus D, and is bisected by the tangent A′B′C′ (§ 9); therefore $O_4$ is on the directrix.

§ 29. *Figure* 22. If a parabola touch four lines, its focus is the point of intersection of the circumscribed circles of the four triangles formed by the four lines, and the Simson line of the complete quadrilateral is the tangent at the vertex of the parabola.

The axis of the parabola is parallel to the line joining the middle points of the diagonals of the quadrilateral. The directrix is the line of the orthocentres of the component triangles.

§ 30. (*a*) Two Simson lines at right angles to one another are the asymptotes of an equilateral hyperbola passing through the vertices of the triangle. By § 18, Cor.

(*b*) The locus of the centres of equilateral hyperbolas circumscribed to a triangle is the nine-point circle.

For the asymptotes, the two Simson lines at right angles to each other, intersect on the nine-point circle. (§ 18.)

(*c*) An equilateral hyperbola circumscribed to a triangle passes through its orthocentre.

For, by § 14, the asymptotes of the equilateral hyperbola circumscribed to ABC are also the asymptotes of that round $BCO_4$; but a hyperbola is determined if its asymptotes and one point on the curve be given; hence the proposition.

§ 31. *Figure* 24. The point where D′(ABC) touches its envelop is found by producing Q′P′ to T so that P′T = Q′P′.

For as D′ and D″ approach indefinitely near, the angle T is indefinitely diminished. P′Q′ and P″Q″ in the limit are parallel, and P′P″ and Q′Q″ are antiparallel.

In the triangles TP′P″, TQ′Q″ we have in the limit

$$\frac{TP'}{TQ''} = \frac{\text{chord } P'P''}{\text{chord } Q'Q''} = \frac{\text{arc } P'P''}{\text{arc } Q'Q''} = \tfrac{1}{2}.$$

But $$\frac{TP'}{TQ''} \text{ in the limit} = \frac{TP'}{TQ'} = \tfrac{1}{2};$$

therefore TP′ = P′Q′, and T is the point where D′(ABC) touches its envelop.

§ 32. *Figure* 27. The envelop of the Simson line is a three-cusped hypocycloid.

Let QPT be the Simson line, T the point of contact with its envelop. Draw a circle PTR through T so as to touch the nine-point circle PQS at P. Since QP = PT the circles are of equal radius, and OR = 3OP. As T traces out its curve it moves at every instant in the direction of the tangent TQ that is perpendicular to RT, and so that TP = PQ, and R moves round the circle ARB. These conditions will be satisfied if T be a fixed point on the circle PTR rolling within the circle ARB which is of triple radius; the curve thus traced is a hypocycloid of three cusps. §§ 30, 31, 32, *Journal de Mathématiques Spéciales*, 2nd Series, Vol. III. pp. 14, 15, 31 (Weill).

§ 33. *Figure* 28. The following are some of the properties of the three-cusped hypocycloid deduced by considering it as the envelop of the Simson line.

(1) If $T_1T_2$ be a tangent at T, meeting the curve again in $T_1$ and $T_2$; $T_1T_2$ is of constant length, and tangents at $T_1$ and T intersect at right angles on the inscribed circle.

For if $Q_1T_1$ and $Q_1T_2$ be constructed Simson lines at right angles, they intersect at $Q_1$ on the nine-point circle of the triangle of reference (the inscribed circle of the hypocycloid); and since they meet their envelop at points such that $Q_1P_1 = P_1T_1$ and $Q_1P_2 = P_2T_2$, therefore $T_1T_2$ is double of $P_1P_2$, that is double of the diameter of the inscribed circle.

Since the sides and perpendiculars of a triangle are Simson lines with respect to that triangle, therefore $T_1T_2$ is a Simson line and touches the hypocycloid; and $QQ_1$, which is the perpendicular at Q, is also a tangent to the curve. (For the inscribed circle is the nine-point circle of the triangle $T_1T_2Q_1$.)

(2) P is the middle point of $T_1T_2$. Therefore any tangent to a hypocycloid meets the inscribed circle in two points, one of which bisects the tangent; and if a perpendicular be drawn to the tangent at the other point, the perpendicular is also a tangent to the hypocycloid.

(3) Since P is the middle point of TQ, therefore $QT_1 = TT_2$. That is, the part of the tangent intercepted between one of the branches of the curve and the inscribed circle is equal to the part between the other two branches.

(4) If $SS_1$ be drawn a tangent to the inscribed circle at $Q_1$, the angles $QQ_1S$, $QQ_1S_1$ are bisected by the tangents to the hypocycloid $Q_1P_2$ and $Q_1P_1$.

For since the triangle $QQ_1T_2$ is right angled, and $P_2$ is the middle point of the hypotenuse, therefore $\angle P_2QQ_1$ is equal to $\angle P_2Q_1Q$. But $\angle P_2QQ_1$ is the angle in the alternate segment, and is therefore equal to $\angle P_2Q_1S$. Since $\angle P_2Q_1P_1$ is a right angle, therefore $\angle QQ_1S_1$ is bisected by $Q_1P_1$.

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## *Seventh Meeting, May 8th* 1885.

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A. J. G. BARCLAY, Esq., M.A., President, in the Chair.

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## On certain formulae for Repeated Differentiation.

### By Professor CHRYSTAL.

In many questions of analysis relating to the theory of plane curves, it is convenient to be able to obtain quickly the expansions of $\left(\frac{d}{dx}\right)^m(y^n)$, and of $\left(\frac{d}{dx}\right)^m(x^py^n)$. These may be obtained as follows:—

By an obvious extension of the theorem of Leibnitz we have

$$\left(\frac{d}{dx}\right)^m(y_1y_2\ldots y_n) = (d_1+d_2+\ldots+d_n)^m y_1y_2\ldots y_n,$$

where $d_1, d_2, \ldots d_n$ differentiate $y_1, y_2, \ldots y_n$ respectively.

Hence by the multinomial theorem

$$\left(\frac{d}{dx}\right)m(y_1y_2\ldots y_n) = m!\Sigma\left(\frac{y_{1r_1}y_{2r_2}\ldots y_{nr_n}}{r_1!\,r_2!\ldots r_n!}\right),$$

where $r_1 \nless 0 \ldots r_n \nless 0$; $r_1 + r_2 \ldots + r_n = n$.

Now in this formula let $y_1 = y_2 \ldots = y_n$, each $= y$, and observe that the term $\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\ldots}$ will occur as often as there are permutations of $n$ things taken all together, $\rho_1$ of which are all alike, $\rho_2$ all alike, &c.; that is $n!/\rho_1!\rho_2!\ldots\ldots$ times; we then obtain

$$\left(\frac{d}{dx}\right)^m(y^n) = m!n!\Sigma\left(\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\rho_1!\rho_2!\ldots\ldots}\right);$$

where
$$r_1 \nless 0 \ngtr m,\ r_2 \nless 0 \ngtr m, \ldots\ldots;$$
$$\rho_1 \nless 1 \ngtr n,\ \rho_2 \nless 1 \ngtr n, \ldots\ldots;$$
$$r_1\rho_1 + r_2\rho_2 + \ldots\ldots = m;$$
$$\rho_1 + \rho_2 + \ldots\ldots = n.$$

Similarly from

$\left(\frac{d}{dx}\right)^m(x^py_1y_2\ldots y_n) = (d+d_1+d_2\ldots+d_n)^m(x^py_1y_2\ldots y_n)$ we derive

$$\left(\frac{d}{dx}\right)^m(x^p y^n)$$

$$= m!\,\Sigma\left((n-r)!\,\frac{p(p-1)\ldots(p-r+1)}{r!}x^{p-r}\,\frac{(y_{r_1})^{\rho_1}(y_{r_2})^{\rho_2}\ldots\ldots}{(r_1!)^{\rho_1}(r_2!)^{\rho_2}\ldots\rho_1!\rho_2!\ldots\ldots}\right);$$

where
$$r \nless 0 \ngtr m,\ r_1 \nless 0 \ngtr m, \ldots\ldots;$$
$$\rho_1 \nless 1 \ngtr n,\ \rho_2 \nless 1 \ngtr n, \ldots\ldots;$$
$$r + \rho_1 r_1 + \rho_2 r_2 + \ldots\ldots = m;$$
$$\rho_1 + \rho_2 + \ldots\ldots = n.$$

Example

$$\left(\frac{d}{dx}\right)^9(y^3) = 9!3!\left[\frac{y_9y^2}{9!2!} + \frac{y_8y_1y}{8!} + \frac{y_7y_1^2}{7!2!} + \frac{y_7y_2y}{7!2!} + \frac{y_6y_3y}{6!3!} + \frac{y_6y_2y_1}{6!2!} + \frac{y_5y_4y}{5!4!}\right.$$
$$\left. + \frac{y_5y_3y_1}{5!3!} + \frac{y_5y_2^2}{5!(2!)^3} + \frac{y_4^2y_1}{(4!)^22!} + \frac{y_4y_3y_2}{4!3!2!} + \frac{y_3^3}{(3!)^4}\right]$$
$$= 3y_9y^2 + 54y_8y_1y + 216y_7y_1^2 + 216y_7y_2y + 504y_6y_3y_1 + 1512y_6y_2y_1$$
$$+ 756y_5y_4y + 3024y_5y_3y_1 + 2268y_5y_2^2 + 1890y_4^2y_1 + 7560y_4y_3y_2 + 1680y_3^3.$$

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## On a method for obtaining the differential equation to an Algebraical Curve.

By Professor CHRYSTAL.

1. Consider the conic represented by the general equation

$$a_0 + b_0x + b_1y + c_0x^2 + c_1xy + c_2y^2 = 0. \quad \ldots \quad \ldots \quad (1)$$

Differentiating three times with respect to $a$ we get

$$b_1(y)_3 + c_2(y^2)_3 + c_1(xy)_3 = 0 \ldots \quad \ldots \quad \ldots \quad (2)$$

where $(y)_3$ stands for $\left(\frac{d}{dx}\right)^3(y)$.

Again, from (2) by successive differentiation we derive

$$b_1(y)_4 + c_2(y^2)_4 + c_1(xy)_4 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

$$b_1(y)_5 + c_2(y^2)_5 + c_1(xy)_5 = 0 \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

From (2) (3) (4), eliminating the remaining constants, we have

$$\begin{vmatrix} (y)_3 & (y^2)_3 & (xy)_3 \\ (y)_4 & (y^2)_4 & (xy)_4 \\ (y)_5 & (y^2)_5 & (xy)_5 \end{vmatrix} = 0 \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

which is one form of the differential equation to the conic (1).

Since
$$(xy)_3 = x(y)_3 + 3(y)_2,$$
$$(xy)_4 = x(y)_4 + 4(y)_3,$$
$$(xy)_5 = x(y)_5 + 5(y)_4,$$
we have rejecting redundant columns, and dropping brackets where they are no longer necessary

$$\begin{vmatrix} y_3 & 3y_2 & (y^2)_3 \\ y_4 & 4y_3 & (y^2)_4 \\ y_5 & 5y_4 & (y^2)_5 \end{vmatrix} = 0 \; \dots \quad \dots \quad \dots \quad \dots \quad (7)$$

From which it already appears that the differential equation to the conic (1) does not contain the independent variable explicitly, and that it contains the highest differential coefficient $y_5$ in the first power only.

We may simplify (7) still further, for

$$\begin{vmatrix} y_3 & 3y_2 & (y^2)_3 \\ y_4 & 4y_3 & (y^2)_4 \\ y_5 & 5y_4 & (y^2)_5 \end{vmatrix} = \begin{vmatrix} y_3 & 3y_2 & 2yy_3 + \;\; 6y_1y_2 \\ y_4 & 4y_3 & 2yy_4 + \;\; 8y_1y_3 + \;\; 6y_2^2 \\ y_5 & 5y_4 & 2yy_5 + 10y_1y_4 + 20y_2y_3 \end{vmatrix} = 2y_2 \begin{vmatrix} y_3 & 3y_2 & 0 \\ y_4 & 4y_3 & 3y_2 \\ y_5 & 5y_4 & 10y_3 \end{vmatrix} = 0.$$

Whence we get immediately

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0 \; \dots \quad \dots \quad \dots \quad (8)$$

The result and the process by which it has been obtained may be compared with Halphen's method (*Jordan Cours d'Analyse de l'Ecole Polytechnique*, t. i., § 53).

2. The method above applied to the general equation of the second degree, and some of the results are of a general character. This will perhaps be best seen by considering the general cubic, whose equation, for present convenience, I write as follows—

$$a_0 + a_1x + a_2x^2 + a_3x^3 + b_1y + b_2y^2 + b_3y^3 + c_1xy + c_2x^2y + d_1xy^2 = 0 \; \dots \; (9)$$

From (9) we derive at once

$$b_1(y)_4 + b_2(y^2)_4 + b_3(y^3)_4 + c_1(xy)_4 + c_2(x^2y)_4 + d_1(xy^2)_4 = 0 \quad \dots \quad (10)$$

From (10) by five successive differentiations we obtain five more equations, and using these along with (10), we eliminate the six constants, and obtain

$$\begin{vmatrix} (y)_4 & (y^2)_4 & (y^3)_4 & (xy)_4 & (x^2y)_4 & (xy^2)_4 \\ (y)_5 & (y^2)_5 & (y^3)_5 & (xy)_5 & (x^2y)_5 & (xy^2)_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ (y)_9 & (y^2)_9 & (y^3)_9 & (xy)_9 & (x^2y)_9 & (xy^2)_9 \end{vmatrix} = 0 \quad \dots \quad \dots \quad (11)$$

Now, observing that, if $(n,1)$, $(n,2)$, $(n,3)$, &c., denote the binomial coefficients of the $n^{\text{th}}$ power, we have

$$\begin{array}{lll}
(xy)_4 = x(y)_4 + (4,1).1.(y)_3, & (x^2y)_4 = x^2(y)_4 + (4,1).2.x(y)_3 + (4,2).2.1.(y)_2, & (xy^2)_4 = x(y^2)_4 + (4,1).1.(y^2)_3, \\
(xy)_5 = x(y)_5 + (5,1).1.(y)_4, & (x^2y)_5 = x^2(y)_5 + (5,1).2.x(y)_4 + (5,2).2.1.(y)_3, & (xy^2)_5 = x(y^2)_5 + (5,1).1.(y^2)_4, \\
\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\
(xy)_9 = x(y)_9 + (9,1).1.(y)_8, & (x^2y)_9 = x^2(y)_9 + (9,1).2.x(y)_8 + (9,2).2.1.(y)_7, & (xy^2)_9 = x(y^2)_9 + (9,1).1.(y^2)_8.
\end{array}$$

Hence rejecting what are obviously redundant columns, the determinantal equation (11) reduces to

$$\begin{vmatrix}
(y)_4 & (y^2)_4 & (y^3)_4 & (4,1)(y)_3 & (4,2)(y)_2 & (4,1)(y^2)_3 \\
(y)_5 & (y^2)_5 & (y^3)_5 & (5,1)(y)_4 & (5,2)(y)_3 & (5,1)(y^2)_4 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(y)_9 & (y^2)_9 & (y^3)_9 & (9,1)(y)_8 & (9,2)(y)_7 & (9,1)(y^2)_8
\end{vmatrix} = 0 \quad \ldots \quad (12)$$

Equation (12) may be further simplified, but it is interesting to remark that here again we have *a differential equation, of the* 9th *order linear in the highest differential coefficient and not explicitly containing the independant variable*, which must be satisfied by the ordinate of every cubic curve.

3. Proceeding exactly as in § 2, if $y$ be determined in terms of $x$ by a *complete* rational integral equation of the $n^{\text{th}}$ degree in $x$ and $y$, and if $p = \frac{1}{2}n(n+3)$, then $y$ satisfies the differential equation

$$\begin{vmatrix}
(y)_{n+1} & (y^2)_{n+1} \cdots & (y^n)_{n+1} & (n+1,1)(y)_n \cdots & (n+1,n-1)(y)_2 & (n+1,1)(y^2)_n \cdots & (n+1,n-2)(y^2)_3 \cdots & (n+1,1)(y^{n-1})_n \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
(y)_p & (y^2)_p \cdots & (y^n)_p & (p,1)(y)_{p-1} \cdots & (p,n-1)(y)_{p-n+2} & (p,1)(y^2)_{p-1} \cdots & (p,n-2)(y^2)_{p-n+1} \cdots & (p,1)(y^{n-1})_p
\end{vmatrix} = 0 \ldots (13);$$

*an equation of the $\frac{1}{2}n(n+3)^{th}$ order, linear in the highest differential coefficient, and not explicitly containing the independent variable.*

4. Adhering still to the supposition that all the coefficients of the primitive equation are independent, it is interesting to notice that the independent variable will not appear in the differential equation even if terms be omitted, provided that in those retained the powers of $x$ which multiply the respective powers of $y$ all occur in order without intermediate omissions. This is obvious on looking at § (2) and observing the reason for the disappearance of the redundant columns.

For example, let us take the cubic

$$a_0 + a_1x + a_2x^2 + (b_0 + b_1x + b_2x^2)y = 0.$$

The resulting differential equation is

$$\begin{vmatrix} (y)_3 & (xy)_3 & (x^2y)_3 \\ (y)_4 & (xy)_4 & (x^2y)_4 \\ (y)_5 & (xy)_5 & (x^2y)_5 \end{vmatrix} = 0,$$

which reduces successively to

$$\begin{vmatrix} (y)_3 & 3(y)_2 & 3.2.x(y)_2 + 3.2.(y)_1 \\ (y)_4 & 4(y)_3 & 4.2.x(y)_3 + 6.2.(y)_2 \\ (y)_5 & 5(y)_4 & 5.2.x(y)_4 + 10.2.(y)_3 \end{vmatrix} = 0, \quad \begin{vmatrix} y_3 & 3y_2 & 3y_1 \\ y_4 & 4y_3 & 6y_2 \\ y_5 & 5y_4 & 10y_3 \end{vmatrix} = 0 \ldots (14)$$

It is interesting to observe that (14) is a differential equation whose complete primitive is a quadratic rational fraction of the most general kind.

A similar equation could of course be obtained for a rational fraction of the most general kind, whose numerator and denominator are of the $m^{th}$ and $n^{th}$ degrees.

5. If the condition in § 4 be not fulfilled the differential equation may contain $x$.

For example,

$$a_0 + a_2x^2 + b_1xy + c_0y^2 = 0$$

gives

$$\begin{vmatrix} (x^2)_1 & (xy)_1 & (y^2)_1 \\ (x^2)_2 & (xy)_2 & (y^2)_2 \\ (x^2)_3 & (xy)_3 & (y^2)_3 \end{vmatrix} = 0,$$

$$\begin{vmatrix} 2x & xy_1 + y & 2yy_1 \\ 2 & xy_2 + 2y_1 & 2yy_2 + 2y_1^2 \\ 0 & xy_3 + 3y_2 & 2yy_3 + 6y_1y_2 \end{vmatrix} = 0,$$

whence $(y^2 - 2xyy_1 + x^2y_1^2)y_3 + 3x(y - xy_1)y_2^2 = 0,$

or throwing out the factor $y - xy_1$

$$(y - xy_1)y_3 + 3xy_2^2 = 0. \ldots \quad \ldots \quad \ldots \quad (15)$$

6. When the constants involved in the primitive integral equation are not independent, but subject to particular relations, the above method must of course be modified.

For example, in the case of the circle $a+bx+cy+d(x^2+y^2)=0$, the differential equation is

$$\begin{vmatrix} y_2 & (x^2+y^2)_2 \\ y_3 & (x^2+y^2)_3 \end{vmatrix} = 0\,; \text{ whence } \begin{vmatrix} y_2 & 2+(y^2)_2 \\ y_3 & (y^2)_3 \end{vmatrix} = 0\,;$$

which gives $-2y_3 + \begin{vmatrix} y_2 & 2yy_2+2y_1{}^2 \\ y_3 & 2yy_3+6y_1y_2 \end{vmatrix} = 0\,;$

that is $$y_3 - y_1 \begin{vmatrix} y_2 & y_1 \\ y_3 & 3y_2 \end{vmatrix} = 0\,;$$

that is $$(1+y_2{}^2)y_3 - 3y_1y_2{}^2 = 0 \ \ldots \qquad \ldots \qquad \ldots \qquad (16)$$

Again, taking the case of the parabola, $a+bx+cy+(dx+ey)^2=0$, we derive

$$\begin{vmatrix} \{(ax+by)^2\}_2 & y_2 \\ \{(ax+by)^2\}_3 & y_3 \end{vmatrix} = 0,$$

whence $$\begin{vmatrix} 2(ax+by)by_2+2(a+by_1)(a+by_1) & y_2 \\ 2(ax+by)by_3+6(a+by_1)by_2 & y_3 \end{vmatrix} = 0\,;$$

that is throwing out redundant rows and factors,

$$\begin{vmatrix} a+by_1 & y_2 \\ 3by_2 & y_3 \end{vmatrix} = 0.$$

This last may be written

$$ay_3 + b \begin{vmatrix} y_1 & y_2 \\ 3y_2 & y_3 \end{vmatrix} = 0.$$

Hence the differential equation to the parabola is

$$\begin{vmatrix} y_3, & \begin{vmatrix} y_1 & y_2 \\ 3y_2 & y_3 \end{vmatrix} \\ y_4, & \begin{vmatrix} y_2 & y_3 \\ 3y_2 & y_3 \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ 3y_3 & y_4 \end{vmatrix} \end{vmatrix} = 0\,;$$

whence $$y_3 \begin{vmatrix} y_2 & y_3 \\ 2y_2 & 0 \end{vmatrix} + y_3 \begin{vmatrix} y_1 & y_2 \\ 3y_3 & y_4 \end{vmatrix} - y_4 \begin{vmatrix} y_1 & y_2 \\ 3y_2 & y_3 \end{vmatrix} = 0\,;$$

that is $$3y_2{}^2y_4 - 5y_2y_3{}^2 = 0,$$

or $$3y_2y_4 - 5y_3{}^2 = 0 \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (17)$$

6. When owing to the omission of certain terms in the general equation, or to particular relations between the constants a differential equation of lower order than that corresponding to the general case is obtained, this differential equation must of course involve the truth of the more general one.

For example, in the case of the parabola we have

$$3y_2y_4 - 5y_3^2 = 0 \qquad \ldots \qquad \ldots \qquad \ldots \qquad (18)$$

whence by differentiation $3y_2y_5 - 7y_3y_4 = 0.$

This may be written $9y_2^2y_5 - 21y_2y_3y_4 = 0$;

or $9y_2^2y_5 - 45y_2y_3y_4 + 24y_2y_3y_4 = 0$;

that is using (18)

$$9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0;$$

which is the general differential equation to a conic section.

## Note on the Integration of $x^m(a+bx^n)^p dx$.

By Thomas Muir, LL.D.

The integration of differentials of the form $x^m(a+bx^n)^p dx$ seems to me to be susceptible of a more methodical mode of treatment than that commonly employed. In the ordinary way of presenting the matter there is little choice left to the student, when such an integration is required of him, between a haphazard, tentative process, and the consultation of a text-book, in which lists of "formulae of reduction" are given.

In beginning the subject with a learner, I should first state that the integration can be made dependent on any one of six different integrals, viz:—

$$(1) \quad \int x^{m-n}(a+bx^n)^p dx,$$

$$(2) \quad \int x^{m+n}(a+bx^n)^p dx,$$

$$(3) \quad \int x^m(a+bx^n)^{p-1} dx,$$

$$(4) \quad \int x^m(a+bx^n)^{p+1} dx,$$

$$(5) \quad \int x^{m-n}(a+bx^n)^{p+1} dx,$$

$$(6) \quad \int x^{m+n}(a+bx^n)^{p-1} dx;$$

that is to say, the integral can be expressed in terms of a like integral in which the index of the monomial factor is greater or less by $n$; in terms of a like integral in which the index of the binomial factor is greater or less by 1; in terms of a like integral in which the index of the monomial factor is less by $n$, and that of the binomial factor greater by 1; and in terms of a like integral in which the index of the monomial factor is greater by $n$, and that of the binomial factor less by 1.

Then I should proceed to show as follows how these transformations could be effected, asking the learner to note particularly that in order to minimize memory work, I had set myself the problem of obtaining them all *by one and the same process.*

(1) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m-n}(a+bx^n)^p dx$.

Take the integral $\int x^{m-n}(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^{m-n}(a+bx^n)^{p+1}dx$$

$$=\frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1}-\int\frac{x^{m-n+1}}{m-n+1}(p+1)(a+bx^n)^p nbx^{n-1}dx,$$

$$=\frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1}-\frac{(p+1)nb}{m-n+1}\int x^m(a+bx^n)^p dx. \qquad (\alpha)$$

Again the fact that $(a+bx^n)^{p+1}=(a+bx^n)^p(a+bx^n)$ gives

$$\int x^{m-n}(a+bx^n)^{p+1}dx \quad =a\int x^{m-n}(a+bx^n)^p dx+b\int x^m(a+bx^n)^p dx. \quad (\beta)$$

Eliminating $\int x^{m-n}(a+bx^n)^{p+1}dx$ between ($\alpha$) and ($\beta$) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m-n}(a+bx^n)^p dx$ as was required.

(2) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m+n}(a+bx^n)^p dx$.

Take the integral $\int x^m(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^{p+1}dx$$

$$=\frac{x^{m+1}}{m+1}(a+bx^n)^{p+1}-\int\frac{x^{m+1}}{m+1}(p+1)(a+bx^n)^p nbx^{n-1}dx, \qquad (\alpha)$$

Again

$$\int x^m(a+bx^n)^{p+1}dx=a\int x^m(a+bx^n)^p dx+b\int x^{m+n}(a+bx^n)^p dx. \quad (\beta)$$

Eliminating $\int x^m(a+bx^n)^{p+1}dx$ between ($\alpha$) and ($\beta$) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m+n}(a+bx^n)^p dx$ as was required.

(3) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p-1}dx$.

Take the integral $\int x^m(a+bx^n)^p dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}}{m+1}(a+bx^n)^p - \int \frac{x^{m+1}}{m+1}p(a+bx^n)^{p-1}nbx^{n-1}dx,$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^p - \frac{pnb}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx. \qquad (\alpha)$$

Again

$$\int x^m(a+bx^n)^p dx = a\int x^m(a+bx^n)^{p-1}dx + b\int x^{m+n}(a+bx^n)^{p-1}dx. \qquad (\beta)$$

Eliminating $\int x^{m+n}(a+bx^n)^{p-1}dx$ between ($\alpha$) and ($\beta$) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p-1}dx$ as was required.

(4) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p+1}dx$.

Take the integral $\int x^m(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^{p+1}dx$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^{p+1} - \int \frac{x^{m+1}}{m+1}(p+1)(a+bx^n)^p nbx^{n-1}dx,$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^{p+1} - \frac{(p+1)nb}{m+1}\int x^{m+n}(a+bx^n)^p dx, \qquad (\alpha)$$

Again

$$\int x^m(a+bx^n)^{p+1}dx = a\int x^m(a+bx^n)^p dx + b\int x^{m+n}(a+bx^n)^p dx. \qquad (\beta)$$

Eliminating $\int x^{m+n}(a+bx^n)^p dx$ between ($\alpha$) and ($\beta$) we have $\int x^m(a+bx^n)^p dx$ in terms of $\int x^m(a+bx^n)^{p+1}dx$ as was required.

(5) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m-n}(a+bx^n)^{p+1}dx$.

Take the integral $\int x^{m-n}(a+bx^n)^{p+1}dx$.

Integrating "by parts" we have

$$\int x^{m-n}(a+bx^n)^{p+1}dx$$

$$= \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \int \frac{x^{m-n+1}}{m-n+1}(p+1)(a+bx^n)^p nbx^{n-1}dx,$$

$$= \frac{x^{m-n+1}}{m-n+1}(a+bx^n)^{p+1} - \frac{(p+1)nb}{m-n+1}\int x^m(a+bx^n)^p dx.$$

Here it is not necessary to proceed further; we have already got what was required.

(6) $\int x^m(a+bx^n)^p dx$ in terms of $\int x^{m+n}(a+bx^n)^{p-1}dx$.

Take the integral $\int x^m(a+bx^n)^p dx$.

Integrating "by parts" we have

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}}{m+1}(a+bx^n)^p - \int \frac{x^{m+1}}{m+1}p(a+bx^n)^{p-1}nbx^{n-1}dx,$$

$$= \frac{x^{m+1}}{m+1}(a+bx^n)^p - \frac{pnb}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx.$$

Here again we have got what was required by one operation only.

The transformations thus being effected attention could be drawn to the leading points of the process by which all the six results are obtained. These are

(*a*) The choice of an integral for operating on.
(*b*) Integrating "by parts."
(*c*) Partition of the integral into two.
(*d*) Elimination of an integral between the two equations thus got.

The only question remaining then to be answered would be as to how the integral we begin with is selected. The answer to this is that the monomial factor is always $x^m$ unless we wish to make the required integral dependent on one where the monomial factor is $x^{m-n}$, in which case we start with $x^{m-n}$; and that the binomial factor is always $(a+bx^n)^{p+1}$ unless we wish to make the required integral dependent on one where the binomial factor is $(a+bx^n)^{p-1}$, in which case we start with $(a+bx^n)^p$. If the learner's memory were not very robust and his spirit did not contemn the aid of a rule in rhyme, I should prefer to dictate to him the answer in a more condensed and winning form, viz:—

Monomial's index leave unchanged,
Except to lessen when required;
Binomial's increase by one,
But change not if 'tis less desired.

This, it seems to me, is Todhunter's third chapter in a nutshell. I am inclined to view it as one of the few unobjectionable outcomes of the modern over-examination system, it having been devised in 1868 when preparing for an approaching hour of trial. Another such product of the same period but of a less artificial character was published in the *Journal of Education* for 1875 with the title "On Integration by Parts." Both of them I have found of considerable service in teaching.

## Historical Note on the so-called Simson line.

By Thomas Muir, LL.D.

The theorem that the feet of the perpendiculars drawn to the sides of a triangle from any point in the circumference of the circumscribing circle are collinear is ascribed (Gerg. Ann. iv., p. 250, ca. 1814) by Servois, though not with confident knowledge, to Simson. Baltzer, who gives us this information, refers also to Gerg. Ann. xiv., p. 28, p. 280, and to Poncelet. Fuller details, showing how the question of authority has hitherto stood, will be found in an extract from a letter of Mr Mackay's in *Nature*, xxx., p. 635. Mr Mackay has further stated that he has not found the property mentioned in any of Simson's published works.

It seems, therefore, of some interest to point out that the theorem is enunciated and proved in a paper with the title "Mathematical Lucubrations," published in Leybourn's *Mathematical Repository*, old series, vol. II., p. 111. The author is Mr William Wallace, assistant mathematical master in Perth Academy, afterwards professor in the Royal Military Academy of Woolwich, and in the University of Edinburgh. The date of publication is 1798. No reference is made to Simson, the theorem apparently being given as new.

Mr MACKAY drew attention to the connection between the theorem of the so-called Simson line, and another theorem rediscovered* by Professor Wallace about 1797, stated in Leybourn's *Mathematical Repository*, old series, vol. I., p. 309, and proved in vol. II., p. 54. The theorem is—If three straight lines touch a parabola, a circle described through their intersections shall pass through the focus of the parabola. Professor Wallace, in his proof, draws perpendiculars from the focus on the three tangents, and shows that the feet of the perpendiculars lie on the tangent at the vertex; in other words, that the tangent at the vertex is the so-called Simson line, which corresponds to the focus.

## Mnemonics for $\pi$, $\frac{1}{\pi}$, $e$.

By J. S. MACKAY, M.A.

The following mnemonics, with one exception, consist of verses or sentences such that if the number of the letters in each word be written down in the order in which the words occur, the desired value will be obtained.

### $\pi$.

The value of $\pi$ to 30 decimal places is got from the quatrain (of whose age and authorship I am ignorant):—

*Que j'aime à faire apprendre un nombre utile aux sages !*
*Immortel Archimède, artiste ingénieur,*
*Qui de ton jugement peut priser la valeur ?*
*Pour moi ton problème eut de pareils avantages.*

In these alexandrine verses the metre and the rhyme are good enough, but the sense is not very brilliant.

Another version consists of only three lines:—

*Que j'aime à faire apprendre un nombre utile aux sages !*
*Glorieux Archimède, artiste ingénieux,*
*Toi de qui Syracuse aime encore la mémoire.*

---

* It was first given in Sectio I., § 15 of I. H. Lambert's *Insigniores Orbitae Cometarum Proprietates*, Augustae Vindelicorum, 1761.

Here the rhyme is at fault, and the last word contains one letter too many. Moreover, the statement about Syracuse cherishing the memory of Archimedes must be understood to be ironical, if one recollects Cicero's account (in the fifth book of his *Tusculan Questions*) of his visit to Archimedes' tomb.

An English couplet, also of unknown authorship, gives the first 13 decimal places:—

*How I wish I could recollect of circle round*
*The exact relation Archimede unwound!*

The "relation Archimede unwound," it may be noted in passing, was that the circumference of a circle was less than $3\frac{10}{70}$ and greater than $3\frac{10}{71}$ of the diameter. (See his *Measurement of the Circle*, Proposition 3.)

## $\frac{1}{\pi}$.

The only French mnemonic for this value which I have seen is a rather forced one, but it may on that account be all the more easily remembered. It is—

*Les* 3 *journées de* 1830 *ont précédé* 89 *à l'envers,*
·3 1830 98

89, that is 1789 (as we say the '45 for 1745), and 1830 are the dates of the first and second French revolutions, and three days sufficed to carry out the second one.

The question—

*Can I discover the reciprocal?*

gives the value correct to six decimal places, ·318310.

## *e.*

This value to 10 decimal places is obtained from

*Tu aideras à rappeler ta quantité à beaucoup de docteurs amis;*

and to 12 decimal places from

*We proffer a mnemonic to remember a standard or neperean base value instantly.*

---

A member gave the following for the values of the roots of the quadratic equation $ax^2 + bx + c = 0$:

*From square of b take* $4ac$;
*Square root extract, and b subtract;*
*Divide by* $2a$; *you've x alway.*

Another mnemonic for the same values, due to Mr N. D. Beatson Bell, is

*When you have written* $-b$,
*The double sign put down;*
*Then* $b^2 - 4ac$
*With square-root mark you crown;*
*Beneath it all a line you trace,*
*Beneath which line* $2a$ *you place.*

The value of the co-efficient of refraction of light in two important cases is got from the following:—

*When rays do pass from air to glass,*
*The value of* $\mu$ *is three by two;*
*But when they pass from air to water,*
*The value of* $\mu$ *is one by three-quarter(s)*!

## *Eighth Meeting, June 12th,* 1885.

THOMAS MUIR, Esq., LL.D., F.R.S.E., in the Chair.

## Summation of certain Series.

### By Professor TAIT.

[*Abstract.**]

The attempt to enumerate the possible distinct forms of knots of any order, though unsuccessful as yet, has led me to a number of curious results, some of which may perhaps be new. The general character of the methods employed will be obvious from an inspection of a few simple cases, and any one who has some practice in algebra may extend the results indefinitely.

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* This Abstract is part of the paper read in June, entitled "On the detection of amphicheiral knots, with special reference to the mathematical processes involved." I have unfortunately mislaid the MS.—P.G.T.

Take, for instance, the series

$$r^m - n(r+s)^m + \frac{n.\overline{n-1}}{1.2}(r+2s)^m - \&c.$$

where the coefficients are the terms of $(1-1)^n$, and the other factors are the $m^{\text{th}}$ powers of the terms of an arithmetical series :—$m$ being a positive integer. The well-known properties of exponential series give us an easy method of summing all expressions of this form. For we have

$$(\epsilon^{px} - \epsilon^{qx})^n = \epsilon^{npx} - n\epsilon^{(\overline{n-1}p+q)x} + \frac{n.\overline{n-1}}{1.2}\epsilon^{(\overline{n-2}p+2q)x} - \&c.$$

which may be written in the form

$$\left((p-q)x + \frac{p^2-q^2}{2!}x^2 + \frac{p^3-q^3}{3!}x^3 + \&c.\right)^n$$

$$= \Sigma\frac{1}{m!}\left(\overline{np}^m - n(np+q-p)^m + \frac{n.\overline{n-1}}{1.2}(np+2\overline{q-p})^m - \&c.\right)$$

Make $np = r$, $q - p = s$; and $p$ and $q$ are known.

The required sum is then the coefficient of $x^m$ in the expansion of

$$m!\left((p-q)x + \frac{p^2-q^2}{2!}x^2 + \ldots\ldots\right)^n.$$

It vanishes therefore, so long as $m \angle n$; and for $m = n$ its value is

$$m!(p-q)^m = (-)^m m! s^m.$$

When the coefficients in the given series are the *alternate* terms of $(1-1)^n$, we have only to treat, as above, the expression

$$(\epsilon^{px} + \epsilon^{qx})^n \pm (\epsilon^{px} - \epsilon^{qx})^n.$$

Such results may be varied *ad libitum*, by introducing two or more quantities in place of $x$, and comparing coefficients of like terms:—*e.g.*, as in finding, by the two methods of expansion, the term in $x^r y^s$ of the quantity

$$(\epsilon^{px} - \epsilon^{qy})^n.$$

But it suffices to have called attention to processes which can give endless varieties of results, some of which may have useful applications.

## *Ninth Meeting, July 10th,* 1885.

A. J. G. Barclay, Esq., M.A., President, in the Chair.

# On the form in Polar co-ordinates of certain expressions occurring in Elastic Solids and in Hydrodynamics.

By Charles Chree, M.A.

In this paper is considered the form taken in Polar co-ordinates by the equivalents of certain well known expressions in Cartesian co-ordinates which occur in Elastic Solids and in Hydrodynamics.

The accompanying figure (fig. 29) will enable us to transform from Cartesians to Polars by simple geometry.

Let the co-ordinates of the point P referred to O, to a fixed plane, and the axis ON be $r$, $\phi$, $\theta$. Let PQSR be an element of surface of a sphere centre O and radius $r$, while P′Q′S′R′ lies on a sphere of radius $r+\delta r$; the several points being taken so that the space between these elements as shown in the figure represents the ordinary polar element of volume. Thus co-ordinates of Q are $r$, $\theta+\delta\theta$, $\phi$

......... R are $r$, $\theta$, $\phi+\delta\phi$.

Let PN and RN be drawn perpendicular to the axis; and let PT, RT be the tangents to the arcs PQ, RS respectively. We are to regard as the *fundamental* directions at any point P the radius vector PP′, the tangent to the arc PQ and the perpendicular to these two, *i.e.*, the tangent to arc PR.

We wish to find the relations between the fundamental directions at P and at the adjacent point S′.

From the figure we see at once that the angle PNR $=\delta\phi$

......... POR $=\sin\theta\delta\phi$

.........PTR $=\cos\theta\delta\phi$.

Thus the following scheme shows at once the direction cosines of the one set of fundamental directions referred to the other—

$$\begin{array}{cc|c|c|c|}
 & & \multicolumn{3}{c}{\overbrace{\hspace{6em}}^{\text{at } S'}} \\
 & & r & \theta & \phi \\
\hline
 & r & 1 & -\delta\theta & -\sin\theta\delta\phi \\
\hline
\text{at P} & \theta & \delta\theta & 1 & -\cos\theta\delta\phi \\
\hline
 & \phi & \sin\theta\delta\phi & \cos\theta\delta\phi & 1 \\
\hline
\end{array}$$

neglecting the squares and products of small quantities of the order $\delta\theta$ or $\delta\phi$.

Regard now the fundamental directions at P as forming a system of fixed rectangular Cartesian co-ordinates to which is referred a given elastic solid.

Let $u$, $v$, $w$ denote the displacements at P along these axes, then the displacements at S′ *along the fundamental directions at* S′ are respectively

$$\left.\begin{array}{l}
u+\dfrac{du}{dr}\delta r+\dfrac{du}{d\theta}\delta\theta+\dfrac{du}{d\phi}\delta\phi \\
v+\dfrac{dv}{dr}\delta r+\dfrac{dv}{d\theta}\delta\theta+\dfrac{dv}{d\phi}\delta\phi \\
w+\dfrac{dw}{dr}\delta r+\dfrac{dw}{d\theta}\delta\theta+\dfrac{dw}{d\phi}\delta\phi
\end{array}\right\} \qquad (1)$$

Now, let U′, V′, W′ denote the displacements at S′ relative to the axes at P, then using the above scheme for the direction cosines and neglecting the squares and products of small quantities we get

$$U'=u+\frac{du}{dr}\delta r+\frac{du}{d\theta}\delta\theta+\frac{du}{d\phi}\delta\phi-v\delta\theta-w\sin\theta\delta\phi$$

or as it may be written

$$U'-u=\frac{du}{dr}\delta r+\left(\frac{1}{r}\frac{du}{d\theta}-\frac{v}{r}\right)r\delta\theta+\left(\frac{du}{r\sin\theta d\phi}-\frac{w}{r}\right)r\sin\theta\delta\phi$$

Similarly we find

$$V'-v=\frac{dv}{dr}\delta r+\left(\frac{1}{r}\frac{dv}{d\theta}+\frac{u}{r}\right)r\delta\theta+\left(\frac{1}{r\sin\theta}\frac{dv}{d\phi}-\frac{w}{r}\cot\theta\right)r\sin\theta\delta\phi$$

$$W'-w=\frac{dw}{dr}\delta r+\frac{dw}{rd\theta}r\delta\theta+\left(\frac{dw}{r\sin\theta d\phi}+\frac{u}{r}+\frac{v}{r}\cot\theta\right)r\sin\theta\delta\phi$$

These may be written thus

$$\begin{aligned}U'-u=\frac{du}{dr}\delta r&+\tfrac{1}{2}\left(\frac{dv}{dr}-\frac{v}{r}+\frac{1}{r}\frac{du}{d\theta}\right)r\delta\theta+\tfrac{1}{2}\left(\frac{dw}{dr}-\frac{w}{r}+\frac{1}{r\sin\theta}\frac{du}{d\phi}\right)r\sin\theta\delta\phi \\
&-\tfrac{1}{2}\left(\frac{dv}{dr}+\frac{v}{r}-\frac{1}{r}\frac{du}{d\theta}\right)r\delta\theta+\tfrac{1}{2}\left(\frac{1}{r\sin\theta}\frac{du}{d\phi}-\frac{dw}{dr}-\frac{w}{r}\right)r\sin\theta\delta\phi \qquad (2)\end{aligned}$$

$$\mathrm{V}' - v = \tfrac{1}{2}\left(\frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta}\right)\delta r + \left(\frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r}\right)r\delta\theta$$

$$+ \tfrac{1}{2}\left(\frac{1}{r\sin\theta}\frac{dv}{d\phi} - \frac{w}{r}\cot\theta + \frac{1}{r}\frac{dw}{d\theta}\right)r\sin\theta\delta\phi.$$

$$- \tfrac{1}{2}\left(\frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)r\sin\theta\delta\phi + \tfrac{1}{2}\left(\frac{dv}{dr} + \frac{v}{r} - \frac{1}{r}\frac{du}{d\theta}\right)\delta r \quad (3)$$

$$\mathrm{W}' - w = \tfrac{1}{2}\left(\frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi}\right)\delta r + \tfrac{1}{2}\left(\frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)r\delta\theta$$

$$+ \left(\frac{1}{r\sin\theta}\frac{dw}{d\phi} + \frac{v}{r}\cot\theta + \frac{u}{r}\right)r\sin\theta\delta\phi$$

$$- \tfrac{1}{2}\left(\frac{1}{r\sin\theta}\frac{du}{d\phi} - \frac{dw}{dr} - \frac{w}{r}\right)\delta r + \tfrac{1}{2}\left(\frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi}\right)r\delta\theta \quad (4)$$

Let us now adopt, for shortness, the following notation

$$\frac{du}{dr} = e\,; \quad \frac{1}{r}\frac{dv}{d\theta} + \frac{u}{r} = f\,; \quad \frac{1}{r\sin\theta}\frac{dw}{d\phi} + \frac{v}{r}\cot\theta + \frac{u}{r} = g \qquad (5)$$

$$\left.\begin{aligned} \frac{dv}{dr} - \frac{v}{r} + \frac{1}{r}\frac{du}{d\theta} &= 2c \\ \frac{dw}{dr} - \frac{w}{r} + \frac{1}{r\sin\theta}\frac{du}{d\phi} &= 2b \\ \frac{1}{r\sin\theta}\frac{dv}{d\phi} + \frac{1}{r}\frac{dw}{d\theta} - \frac{w}{r}\cot\theta &= 2a \end{aligned}\right\} \qquad (6)$$

$$\left.\begin{aligned} \frac{dv}{dr} + \frac{v}{r} - \frac{1}{r}\frac{du}{d\theta} &= 2\zeta \\ \frac{1}{r\sin\theta}\frac{du}{d\phi} - \frac{dw}{dr} - \frac{w}{r} &= 2\eta \\ \frac{1}{r}\frac{dw}{d\theta} + \frac{w}{r}\cot\theta - \frac{1}{r\sin\theta}\frac{dv}{d\phi} &= 2\xi \end{aligned}\right\} \qquad (7)$$

Then the displacements may be written

$$\left.\begin{aligned} \mathrm{U}' - u &= e\delta r + cr\delta\theta + br\sin\theta\delta\phi - \zeta r\delta\theta + \eta r\sin\theta\delta\phi \\ \mathrm{V}' - v &= c\delta r + fr\delta\theta + ar\sin\theta\delta\phi - \xi r\sin\theta\delta\phi + \zeta\delta r \\ \mathrm{W}' - w &= b\delta r + ar\delta\theta + gr\sin\theta\delta\phi - \eta\delta r + \xi r\delta\theta \end{aligned}\right\} \qquad (8)$$

The last two terms in each expression are obviously the displacements due to the rotation of the body as a whole through the elementary angles $\xi$, $\eta$, $\zeta$, about the three fundamental directions through P.

The first three terms in each expression indicate a relative displacement of points surrounding P such that, taking the fundamental directions at P as three Cartesian axes of $x$, $y$, $z$, every point on the quadric $ex^2 + fy^2 + gz^2 + 2ayz + 2bzx + 2cxy = \text{constant}$..................(9) is displaced in the direction of the normal to the quadric.

Since the rotatory displacement first considered causes no relative displacement of the particles of the solid, it introduces no elastic forces or stresses. These stresses, depending as they do on relative displacements, are thus functions only of the six coefficients $e, f, g, a, b, c$, of the above quadric.

This quadric has been called the Elongation Quadric, and its properties are well known.

The stresses must operate in obedience to the principles of Conservation of Energy, and so must be derivable from a potential which is a quadratic function of the relative displacements or strains, the potential representing the work done by the stresses during the corresponding strains. The potential, as representing physical properties, must for an isotropic medium be independent of all systems of axes, and so must consist of such quadratic expressions involving $e, f, g, a, b, c$, as are independent of the axes chosen.

From the elongation quadric we see that such independent quadratic expressions must be derivable from $e+f+g$ and $a^2+b^2+c^2-(ef+eg+fg)$ the two first invariants of the above quadric. Thus for the potential we have an expression of the form $2W = A(e+f+g)^2 + B\{a^2+b^2+c^2-(ef+eg+fg)\}$ where A and B are constants.

This is at once seen to be the same as the form used by * Thomson and Tait, writing $B = 4n$, $A = k + \frac{4}{3}n$, and noticing that their $a$, $b$, and $c$ are double of mine.

Adopting their usual constants $m$ and $n$, and writing $\frac{a}{2}$ for $a$ etc. in the above, we get the most convenient form

$$2W = (m+n)(e+f+g)^2 + n\{a^2+b^2+c^2-4(ef+eg+fg)\}\text{.........}(10)$$

and so W is completely known from (5) and (6) in terms of Polars, noticing now to drop the 2 in the second side of (6).

From W we can at once deduce the elastic forces at each point by regarding the fundamental directions at that point as forming three rectangular axes. Thus using Thomson and Tait's notation the stresses are P, Q, R, S, T, U,

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* (7) of § 695, Part II.

$$\left.\begin{aligned} \text{P being } &= \frac{dw}{de} \text{ etc.,} \\ \text{S} &= \frac{dw}{da} \text{ etc.,} \end{aligned}\right\} \qquad (11)$$

The surface conditions are those given by * Thomson and Tait.

It will be found safer to use $\lambda$, $\mu$, $\nu$, however, for the direction cosines of the normal. It should be noticed that here the normal is referred to the fundamental directions at each point of the surface which vary from point to point. If, however, the surface be spherical $\mu = 0 = \nu$ and $\lambda = 1$, and the conditions are much simplified.

The quantity $\Delta \equiv e + f + g$ occurs frequently; its value is from (5)

$$\begin{aligned} \Delta &\equiv \frac{du}{dr} + \frac{2u}{r} + \frac{1}{r}\frac{dv}{d\theta} + \frac{v}{r}\cot\theta + \frac{1}{r\sin\theta}\frac{dw}{d\phi} \\ &\equiv \frac{1}{r^2}\left\{\frac{d(ur^2)}{dr} + \frac{1}{\sin\theta}\frac{d(vr\sin\theta)}{d\theta} + \frac{1}{\sin^2\theta}\frac{d(wr\sin\theta)}{d\phi}\right\} \end{aligned} \qquad (12)$$

it indicates the expansion of unit volume during the strain.

The values of $\xi$, $\eta$, $\zeta$ in (7) can be written more concisely thus

$$\left.\begin{aligned} 2\zeta &= \frac{1}{r}\left\{\frac{d(vr)}{dr} - \frac{du}{d\theta}\right\} \\ 2\eta &= \frac{1}{r}\left\{\frac{1}{\sin\theta}\frac{du}{d\phi} - \frac{d(wr)}{dr}\right\} \\ 2\xi &= \frac{1}{r\sin\theta}\left\{\frac{d(w\sin\theta}{d\theta} - \frac{dv}{d\phi}\right\} \end{aligned}\right\} \qquad (13)$$

It will be found that the equations given by † Lamé for the equilibrium or motion of an Elastic Solid in Polars become, when the change in notation is allowed for, the following

$$\left.\begin{aligned} (m+n)r^2\sin\theta\frac{d\Delta}{dr} - 2n\frac{d(\zeta r\sin\theta)}{d\theta} + 2n\frac{d(\eta r)}{d\phi} &= \rho r^2\sin\theta\left(\frac{d^2u}{dt^2} - \mathrm{R}\right) \\ (m+n)\sin\theta\frac{d\Delta}{d\theta} - 2n\frac{d\xi}{d\phi} + 2n\frac{d(\zeta r\sin\theta)}{dr} &= \rho\sin\theta\left(\frac{d^2v}{dt^2} - \Theta\right) \\ (m+n)\frac{1}{\sin\theta}\frac{d\Delta}{d\phi} - 2n\frac{d(\eta r)}{dr} + 2n\frac{d\xi}{d\theta} &= \rho\frac{1}{\sin\theta}\left(\frac{d^2w}{dt^2} - \Phi\right) \end{aligned}\right\} \qquad (14)$$

where $\rho$ is the density of the solid, and R, $\Theta$, $\Phi$ are the components, along the fundamental directions at the point considered, of the external forces. Thus the expressions $\xi$, $\eta$, $\zeta$, are of great importance.

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* § 662, (1); § 670, (10); and § 734 of Part II.

† Leçons sur l'Elasticité.

In Hydrodynamics we are to regard $u$, $v$, $w$ as the velocities at any point in the fluid.

The equations (8) then give us the relative velocities of the fluid at two adjacent points along the fundamental directions at the first of the points. The two last terms in each indicate a motion of rotation in adjacent points in virtue of which the element of fluid moves as a solid body. Unless these terms vanish there is vortex motion, and $\xi$, $\eta$, $\zeta$ are the components of the vorticity about the fundamental directions at the point considered.

If $\xi$, $\eta$, $\zeta$ all vanish, we get as before, the quadric (9).

Its hydrodynamical property is that at every point on its surface the fluid is moving along the normal.

This corresponds exactly to the quadric given by * Lamb noticing the difference of notation.

Thus the conditions for irrotational motion are

$$\left.\begin{aligned} \frac{d(vr)}{dr} - \frac{du}{d\theta} &= 0 \\ \frac{1}{\sin\theta}\,\frac{du}{d\phi} - \frac{d(wr)}{dr} &= 0 \\ \frac{d(w\sin\theta)}{d\theta} - \frac{dv}{d\phi} &= 0 \end{aligned}\right\} \qquad (15)$$

The quantity $\Delta$, see (12), is termed the † *expansion* of the fluid; for an incompressible fluid it is zero.

From (13) it is easily found that

$$\frac{d(\xi r^2)}{dr} + \frac{1}{\sin\theta}\,\frac{d(\eta r\sin\theta)}{d\theta} + \frac{1}{\sin^2\theta}\,\frac{d(\zeta r\sin\theta)}{d\phi} = 0 \qquad (16)$$

$\therefore$ $\xi$, $\eta$, $\zeta$ might be the components along the fundamental directions of the velocities of an incompressible fluid.

The conditions (15) may be verified by transferring directly to Polars the ordinary expressions for the components of vorticity, and resolving them about the fundamental directions at the point considered. They indicate that $udr + vrd\theta + wr\sin\theta d\phi$ is a complete differential.

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* See Lamb's "Motion of Fluids," Chap. III.

† *Ibid.*, p. 6.

# Notes on the method of Orthogonal Projection.

By Robert J. Dallas, M.A.

This paper is designed to call attention to certain extensions of this method, which do not seem to be generally recognised. Most writers consider the ellipse alone as amenable to orthogonal projection, whereas all the conic sections are so.

With the hyperbola we may use this method in three ways—

(1) By projecting a given diameter in the figure to be a transverse axis of the projected figure, when the result will follow by symmetry.

(2) By projecting the hyperbola into a rectangular one.

(3) By an *imaginary orthogonal projection* from the circle.

Of the first two, which are obvious enough, I shall give one or two examples, although I think it is scarcely necessary to do so, especially as I believe all are amenable to the third mode of treatment. I shall omit figures for the most part as they can readily be drawn.

I. P′CP is any diameter, P′Q, P′R are drawn parallel to the asymptotes, meeting tangent at P, in Q and R. Show that PQ = PR.

Project the figure so that P′CP becomes the transverse axis, then the latent symmetry of the figure becomes *explicit*, and it is manifest that PQ = PR. So for a multitude of other examples.

II. CP and CD are conjugate; from P a tangent PT is drawn to the conjugate hyperbola, and from D, DQ, to the original hyperbola. Show that CQ and CT are conjugate. (*Aberdeen Senior Mathematical Examination* 1884.)

Project the hyperbola into a rectangular one. Then the whole figure has the asymptote between CP and CD for an axis of symmetry, and if we made it revolve half a revolution round it as an axis, we should get the same figure again. Hence CQ and CT are equally inclined to the asymptote, and by property of rectangular hyperbola are therefore conjugate. Here again, a latent symmetry has been made explicit.

I now pass to the third method, which has the widest scope. Following the *principle of continuity*, we argue that as the ellipse and hyperbola shade into one another by insensible gradations, and are conics of co-ordinate importance, that is, each requires five conditions, therefore propositions which are true for the one remain true for the other. This is reasoning from analogy rather than imaginary projection, but it comes to the same thing. Let it be granted then, that from a circular cylinder we can cut any hyperbola, just as we can cut any ellipse—which of course involves planes inclined at imaginary angles, whose sines are greater than unity. [Throughout this paper I shall use "*concentric*" as an abbreviation for the three attributes,—concentric, similar, and similarly situated.] Now we know that the asymptotes of a hyperbola may themselves be considered a hyperbola, for which the ratio of similitude $=0$. This may easily be shown, for on any radius vector OP take $p$ so that $Op=\mu OP$, then when $\mu$ is very small we get a hyberbola lying very close to the asymptotes, and in the limit coinciding with them. This enables us to prove a well-known proposition, namely, that

III. If the tangent at P meet the asymptotes in Q and R, then triangle CQR has constant area.

For take two concentric circles—then any tangent to the inner, cuts off from the outer circle, a segment of constant area, by symmetry. Areas are projective properties, and we infer that the same truth holds with regard to any two *concentric* conics, whether ellipses or hyperbolas. Hence any tangent to the inner of two *concentric* hyperbolas cuts off a constant area from the outer. And they have the same asymptotes. Now let the outer hyperbola (the one nearer the asymptotes) shrink up till it becomes the asymptotes, then the proposition still remains true, and we get the result we set out to prove, the curvilinear segment degenerating into the triangle CQR. So the facts that QR is bisected in P, and that if we draw any chord RQQ′R′ across the hyperbola and asymptotes, then $RQ=Q'R'$ follow at once from considering two concentric circles. Many other examples might be given. The following, as also Ex. I, can be treated both by this method and by the first.

IV. PQ is any chord; parallels to the asymptotes are drawn through P and Q. Show that the other diagonal of the parallelogram thus formed passes through C.

Let PQ be any chord of a circle. From P and Q as centres, describe two equal circles; their common chord bisects PQ at right angles, and therefore passes through the centre.

Now let the two circles become very small; their points of crossing are now imaginary, but their radical axis or common chord will still pass through the centre. Project the big circle orthogonally into a hyperbola, then the two very small circles or points (as they are in the limit) become pairs of straight lines parallel to the asymptotes. For we have already seen that any pair of straight lines is a very small hyperbola, and is therefore projected into a point, or very small circle. Hence the result follows.

I next proceed to the relation between the hyperbola and its conjugate. Since both are *concentric* with the asymptotes, it follows that they are *concentric* with each other. Hence, as before, we can argue from two concentric circles to the hyperbola and conjugate. It is important to note that two circles are required. For instance, if in the ellipse CP and CD are conjugate, the parallelogram contained by CP, CD, and the tangents at P and D, has constant area. The same thing appears to be the case in the hyperbola, but there is a slight difference—one fact is not the projection of the other. In fact the latter case is the projection of the following—(Fig. 30).

P and D are points on two concentric circles such that CP and CD are at right angles; tangents PR, DR are drawn. Show that rectangle CR has constant area. As it has of course by symmetry.

The ratio of similitude for the hyperbola and conjugate is $\sqrt{-1}$, as may be established thus.

V. Let us have two concentric circles, and let $CP = \mu CD$, and be at right angles to CD always. (Fig. 30). Draw the tangents PR, DR, then $CR^2 = (1 + \mu^2)CD^2$;
therefore R traces out a circle whose radius $= \sqrt{1 + \mu^2}CD$.

Hence for two *concentric* hyperbolas, the ratio of similitude being $\mu$, the locus of intersection of tangents drawn at points where they are met by conjugate diameters, is another *concentric* hyperbola, whose ratio of similitude is $\sqrt{1 + \mu^2}$.

Now for the hyperbola and its conjugate, the locus is the asymptotes—that is

$$\sqrt{1 + \mu^2} = 0$$

therefore $$\mu = \sqrt{-1},$$

the ratio of similitude being 0 for the asymptotes.

As another example take the following.

VI. Show that the hyperbola is its own reciprocal with regard to the conjugate.

From any point P (Fig. 31) in the outer of two concentric circles, draw tangents to the inner. The chord of contact will, by symmetry, touch a concentric circle, and if $CR = \mu CP$, then $CW = \mu^2 CP$.

In the case we are considering, $\mu = \sqrt{-1}$, and therefore we infer that if P be any point on the hyperbola its polar with regard to the conjugate, touches the original hyperbola at P′, the other extremity of diameter CP, since $CP' = -CP = CP \times (\sqrt{-1})^2$.

As a final example, I take one involving focal properties. The projective property of the focus may be readily established geometrically, for we know that

1. A revolving right angle, makes with the circular asymptotes a harmonic pencil, and with no pair of *fixed* lines *but* the circular asymptotes.
2. If OL, OM be tangents to a conic, and the tangent at P meet LM in R, then OL, OP, OM, OR is harmonic.
3. If the tangent at P meet directrix in Z, then PSZ is a right angle. Combining these we see that the tangents to a conic from a focus, must pass through the circular points at infinity.

VII. A number of parabolas pass through a fixed point, and touch two lines. Show that the chord of contact envelopes a hyperbola. (*Aberdeen Senior Mathematical Examination* 1884).

Project orthogonally any hyperbola asymptotic to OP, OQ, (Fig. 32) into a circle, then OP, OQ become the circular asymptotes, and the parabolas remain parabolas, while O becomes their focus S. Hence the problem becomes—Given a point and a focus of a parabola, find the envelope of polar of the focus, or the directrix. And this is of course a circle, whose centre is the fixed point, and which passes through S. Returning to the original figure, we find that the envelope is a *hyperbola whose centre is* A, *which passes through* O, *and whose asymptotes are parallel to* OP, OQ.

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## On a number of concurrent spheres.

By R. E. Allardice, M.A.

This paper included an analytical proof of the following theorem:

If five spheres 1, 2, 3, 4, 5, pass through the same point, and if the four points in which the four spheres 1, 2, 3, 4, intersect in sets of three be coplanar; and if the same be true for the sets 1, 2, 3, 5; 1, 2, 4, 5; 1, 3, 4, 5; it will also be true for the remaining set 2, 3, 4, 5.

The same theorem is true for similar and similarly situated quadric surfaces.

# Theorems connected with three mutually tangent circles.[*]

By THOMAS MUIR, LL.D.

1. The communication on this subject, as originally made to the society, consisted of a series of theorems, giving (1) expressions for the radii of a great many sets of circles, (2) identities connecting several sets of these radii, and (3) miscellaneous identities closely related thereto. As, however, the paper culminated in a general theorem which may be looked upon as fundamental, and the proof of which makes evident the mode of arriving at the said expressions for radii, and as the relations connecting sets of radii are easily found when attention has been directed to their existence, I have thought it best to print little more than the fundamental theorem and a few auxiliary notes.

2. The theorem is—

*If the radii of the two smaller semi-circles of an arbelos*[†] *be* a *and* b, *and a circle be inscribed in the arbelos, then circles in the three curvilineal triangles cut off by the preceding circle, then circles in the nine curvilineal triangles cut off by the three immediately preceding circles, and so on* ad libitum, *the expression for the radius of any one of these circles is of the form*

$$\frac{ab(a+b)}{\xi a^2+\eta ab+\zeta b^2}$$

*where* $\xi$, $\eta$, $\zeta$ *are integers.*

We know that if $r_1$, $r_2$, $r_3$ be the radii of three circles in mutual contact, $r_4$ the radius of one of the circles touching all the said three, and $r_5$ the radius of the other, then

$$\frac{1}{r_4}+\frac{1}{r_5}=\frac{2}{r_1}+\frac{2}{r_2}+\frac{2}{r_3}\,; \qquad \text{(A)}$$

it being understood that when one of the circles *encloses* the four others its radius is made negative. Now, should it happen that $r_1$, $r_2$, $r_3$, and $r_4$ are each of the form specified in the theorem, their reciprocals must have the common denominator $ab\ (a+b)$, and hence from (A) $r_5$ must be of that form likewise. But the radii of the three original circles of the arbelos are of the form specified, for

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[*] This paper was read at the February meeting.

[†] See Mr Mackay's paper, pp. 2-11 and fig. 33.

$$a = \frac{ab(a+b)}{0.a^2 + 1.ab + 1.b^2},$$

$$b = \frac{ab(a+b)}{1.a^2 + 1.ab + 0.b^2},$$

$$\text{and } a + b = \frac{ab(a+b)}{0.a^2 + 1.ab + 0.b^2};$$

and it is known that the radius of the first inscribed circle is so also, being

$$= \frac{ab(a+b)}{a^2 + ab + b^2};$$

hence each of the three next radii has the specified form. The same then follows regarding each of the nine radii succeeding these, and so on. The theorem is thus established.

3. In the proof we have assumed that the radius of the first inscribed circle is known. It is not necessary, however, to do so; for the formula (A) suffices to find it.

The three circles in mutual contact are in this case the three original circles of the arbelos, with the radii $a$, $b$, $a+b$; $r_4$ is unknown, but we see from symmetry that it is the same as $r_5$; hence from (A) we have

$$\frac{1}{r_5} + \frac{1}{r_5} = \frac{2}{a} + \frac{2}{b} - \frac{2}{a+b},$$

$$\text{and } \therefore \frac{1}{r_5} = \frac{a^2 + ab + b^2}{ab(a+b)},$$

as was to be shown.

4. If we denote the reciprocal of the radius

$$\frac{ab(a+b)}{\xi a^2 + \eta ab + \zeta b^2}$$

by $(\xi, \eta, \zeta)$, such calculations of radii can be made with great rapidity. Turning for a moment to the diagram (fig. 33), where, be it observed, the reciprocal of the radius of any circle is given in this notation at the centre, let us calculate the radius of the circle PQR. PQR touches the three mutually tangent circles (1, 1, 0), (1, 1, 1), (0, 1, 0), and the other circle which does so is the circle (0, 1, 1); hence

$$\begin{aligned}(\xi, \eta, \zeta) + (0, 1, 1) &= 2\{(1, 1, 0) + (1, 1, 1) - (0, 1, 0)\},\\ &= 2\{(2, 2, 1) - (0, 1, 0)\},\\ &= (4, 2, 2);\\ \therefore \quad (\xi, \eta, \zeta) &= (4, 1, 1),\end{aligned}$$

as appears on the diagram.

5. The radius of the $n^{\text{th}}$ circle of any particular set of circles which have properties in common is readily obtainable. The following sets are noteworthy; even the few members of them which appear in the diagram enable one to tell the radius of the $n^{\text{th}}$ member.

$(0, 1, 1), (1, 1, 1), (4, 1, 1), (9, 1, 1), \ldots\ldots\ldots (n^2, 1, 1)$
$(0, 1, 1), (1, 1, 4), (4, 1, 9), (9, 1, 16), \ldots\ldots\ldots (n^2, 1, \overline{n+1}^2)$
$(0, 1, 1), (1, 1, 9), (4, 1, 25), (9, 1, 49), \ldots\ldots (n^2, 1, \overline{2n+1}^2)$
$(0, 1, 1), (1, 1, 16), (4, 1, 49), \ldots\ldots\ldots\ldots\ldots\ldots (n^2, 1, \overline{3n+1}^2)$

……

The circles of the first row extend up the horn whose tip is at A, those of the second row up a horn whose tip is at G, and so on.

Similarly we have the series

$(1, 1, 0), (1, 1, 1), (1, 1, 4), (1, 1, 9), (1, 1, 16), \ldots (1, 1, n^2)$
$(1, 1, 0), (4, 1, 1), (9, 1, 4), (16, 1, 9), \ldots\ldots\ldots\ldots (\overline{n+1}^2, 1, n^2)$
$(1, 1, 0), (9, 1, 1), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\overline{2n+1}^2, 1, n^2)$

……

where every triad of coefficients is got by reversing the order of the corresponding triad in the previous sets, *i.e.*, as we should expect, by interchanging $a$ and $b$.

Next we have the series which extends up the horn whose tip is C, viz.:—

$(1, 1, 1), (4, 7, 4), (9, 17, 9), \ldots\ldots\ldots\ldots\ldots\ldots (n^2, 2n^2 - 1, n^2).$

Then there are the lateral series

$(4, 1, 1), (4, 1, 9), (4, 1, 25), \ldots\ldots\ldots\ldots\ldots\ldots (4, 1, \overline{2n-1}^2)$
$(4, 7, 4), (4, 7, 12), (4, 7, 28), \ldots\ldots\ldots\ldots\ldots\ldots (4, 7, 4n^2 - 4n + 4).$
&c., &c.

6. Although in enunciating the theorem of § 2 I have spoken of the arbelos, it should be carefully noted that what we have got to do with is properly not semicircles but circles. We start with one circle (0, 1, 1) touching another (0, 1, 0) internally, then in the space between the two circumferences, and touching them both, we describe a series of circles, each one of which touches the one preceding it and the one following it. Now, there are two such series which can be symmetrically situated with respect to the common diameter AB; first,......the series, (1, 1, 4), (1, 1, 1), (1, 1, 0), (1, 1, 1), (1, 1, 4),.......the

largest of which has AC for diameter; second, the series of which the two largest have AC for a tangent. (See fig. 9.)

If this latter series be described, and circles inscribed in the curvilineal triangles thereby formed, and so on, as before, the radius of every one of these circles likewise can be expressed in the form $ab(a+b)/(\xi a^2+\eta ab+\zeta b^2)$, $\zeta$ being now, however, sometimes fractional.

The main series, corresponding to the series (1, 1, 0), (1, 1, 1), ... in the former diagram, is

$$(1,\ 1,\ \tfrac{1}{4}),\ (1,\ 1,\ \tfrac{9}{4}),\ (1,\ 1,\ \tfrac{25}{4}),\ldots\ldots\ldots$$

Each of the latter, it will be seen, falls between two of the former, the two series dovetailing into one with a common law of progression, viz. :—

$$(1,\ 1,\ 0),\ (1,\ 1,\ (\tfrac{1}{2})^2),\ (1,\ 1,\ 1^2),\ (1,\ 1,\ (\tfrac{3}{2})^2),\ (1,\ 1,\ 2^2),\ldots\ldots\ldots$$

Further, the two new lateral series are

$$(4,\ 1,\ 0),\ (4,\ 1,\ 4),\ (4,\ 1,\ 16),\ldots\ldots\ldots$$
$$(4,\ 7,\ 7),\ (4,\ 7,\ 19),\ (4,\ 7,\ 39),\ldots\ldots\ldots$$

Each of these also dovetails with the analogous series in the former diagram, the pair of combined series being

$$(4,\ 1,\ 0),\ (4,\ 1,\ 1),\ (4,\ 1,\ 4),\ (4,\ 1,\ 9),\ldots\ldots(4,\ 1,\ n^2);$$
$$(4,\ 7,\ 4),\ (4,\ 7,\ 7),\ (4,\ 7,\ 12),\ (4,\ 7,\ 19),\ldots(4,\ 7,\ n^2+2n+4).$$

If the one diagram be superposed on the other so that the two can be viewed as one, these and other ties of relationship are well illustrated. For example, it will also be seen that any corresponding pair of circles taken from the two lateral series of the one diagram, touch the two original circles (0, 1, 1), (0, 1, 0) in the points where a circle of the main series of the other diagram touches them.

But the field of such curiosities is unlimited.

Fig. 1

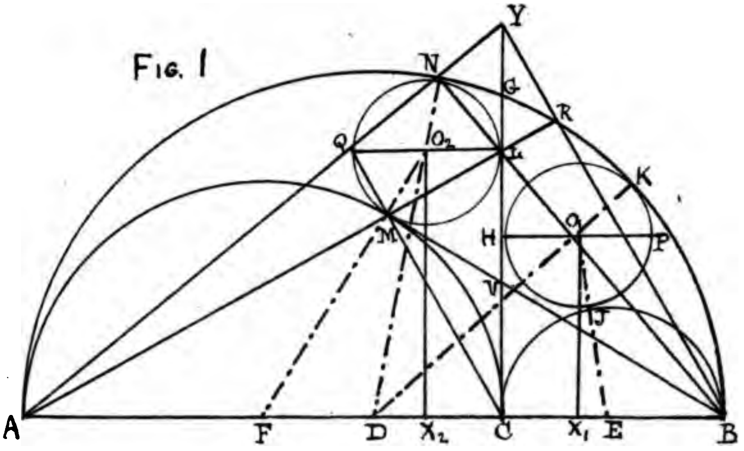


Fig. 2

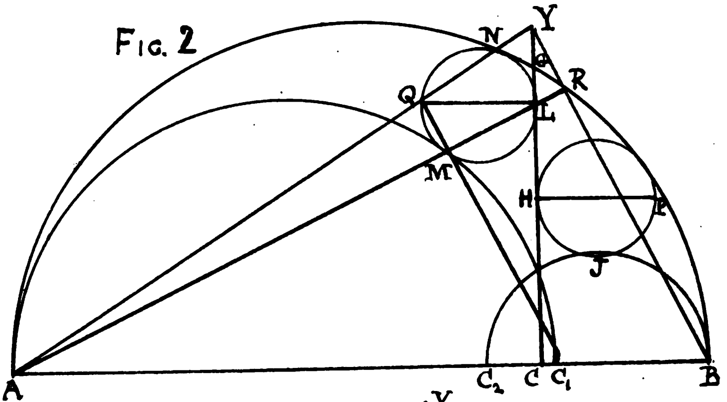


Fig. 3

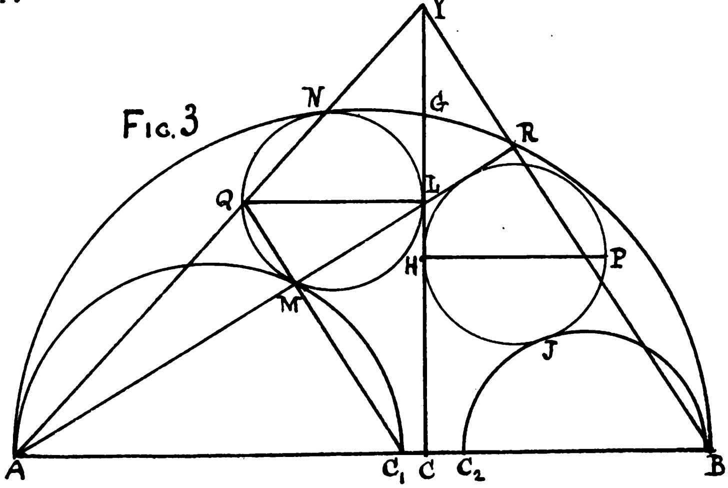
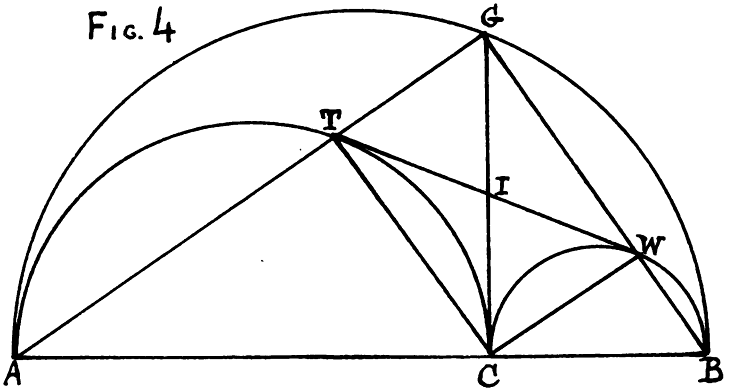


FIG. 4



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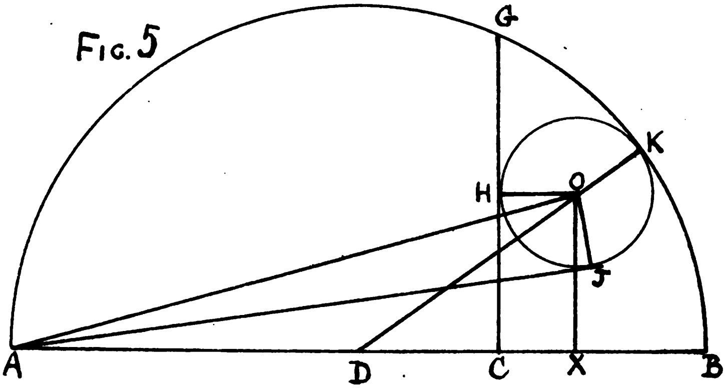

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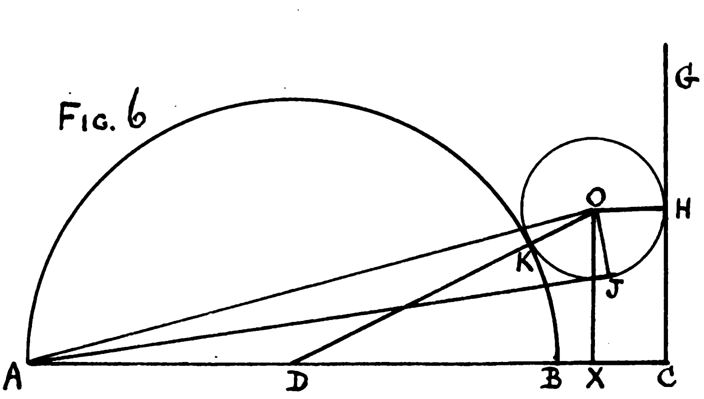


Fig. 7

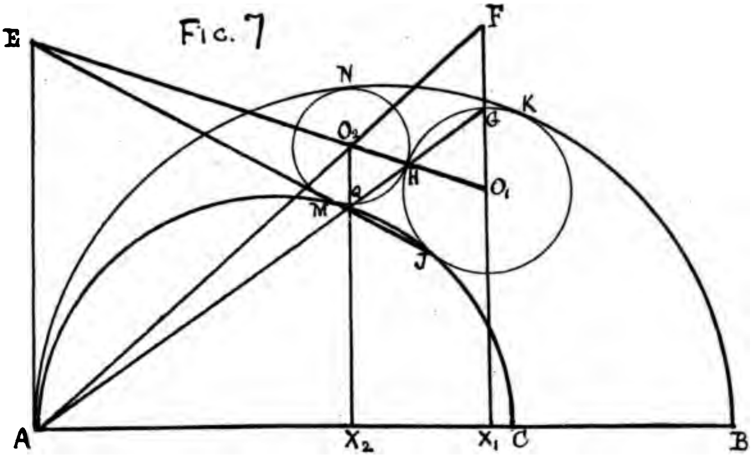


Fig. 8

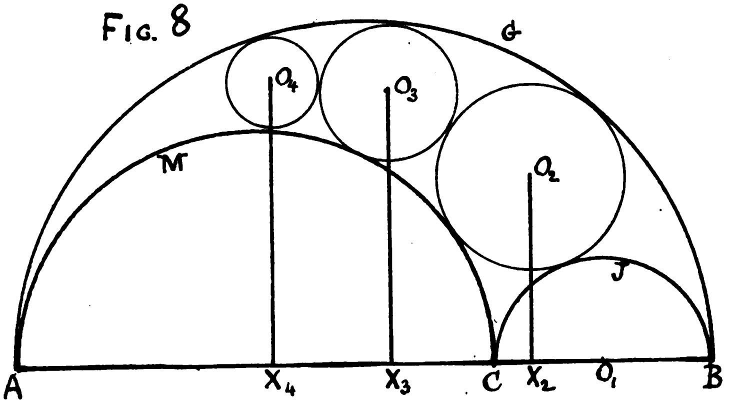


Fig. 9

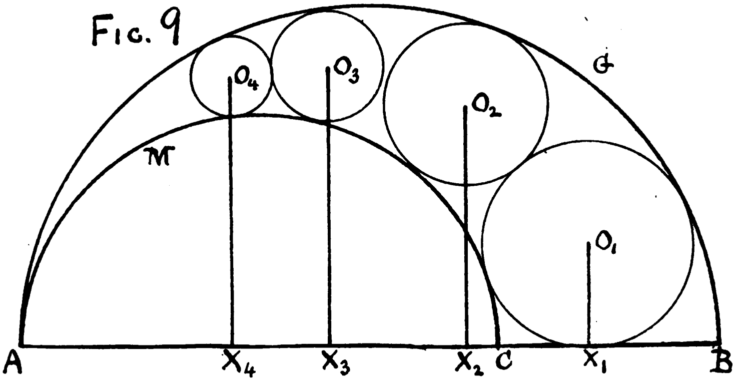
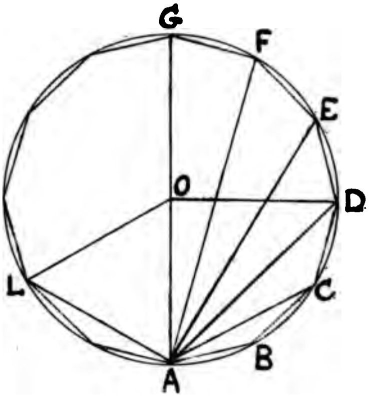


FIG. 10



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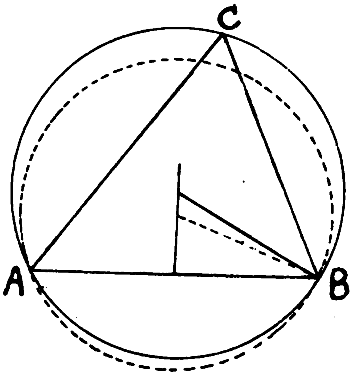

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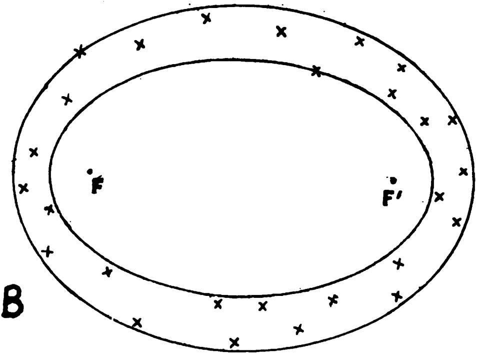

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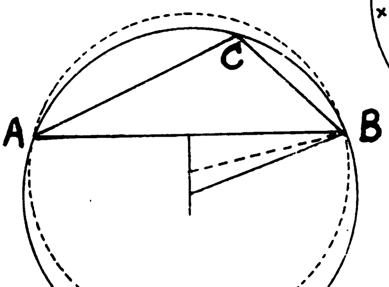


FIG. 14

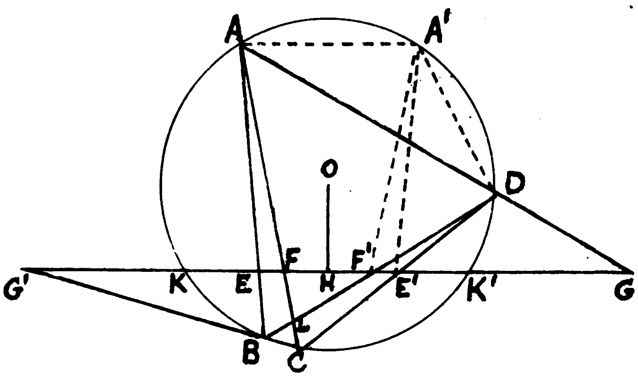


Fig. 16

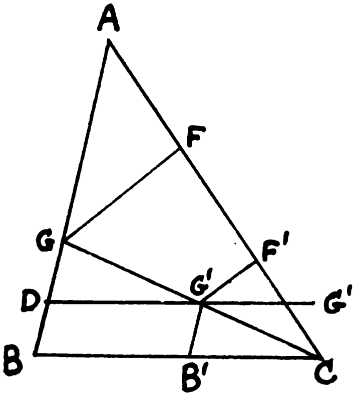


Fig. 17

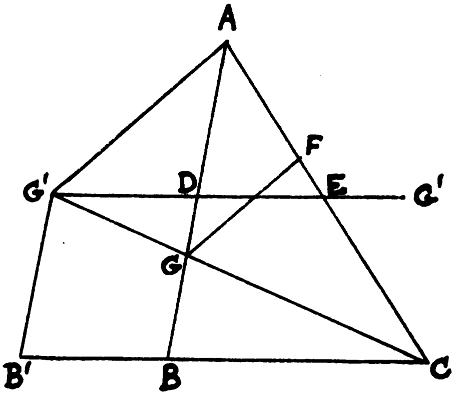


Fig. 18

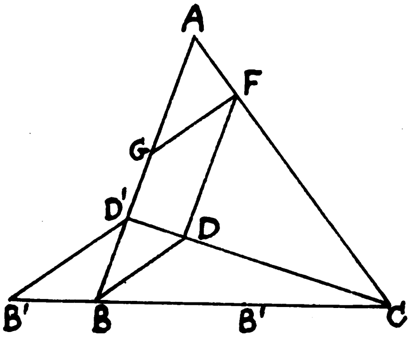


Fig. 19

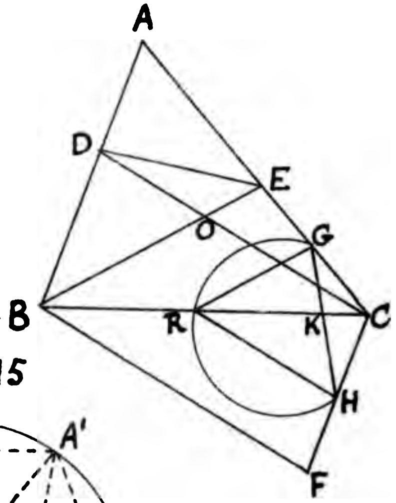


Fig. 15

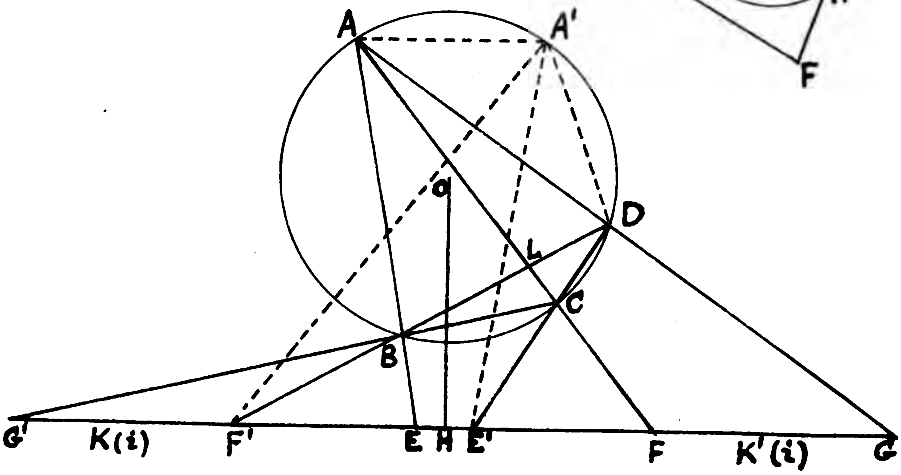


FIG. 16

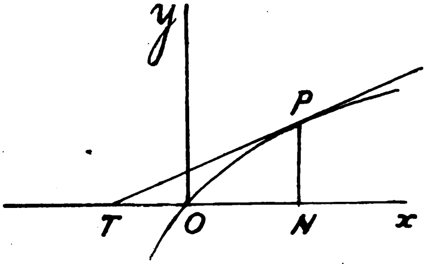
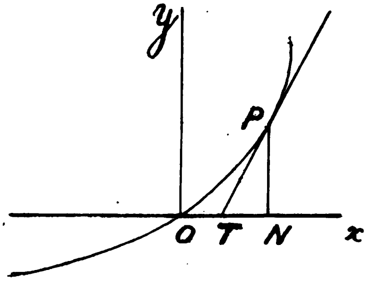


FIG. 17



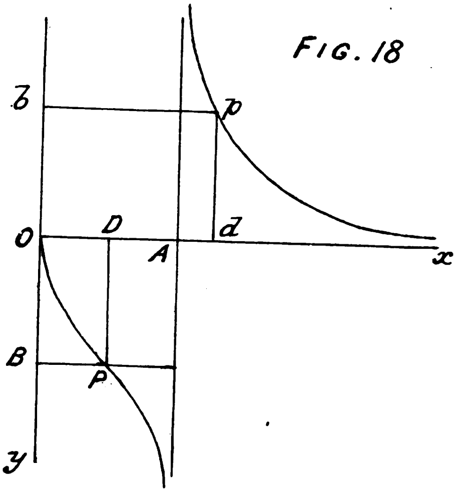


FIG. 18

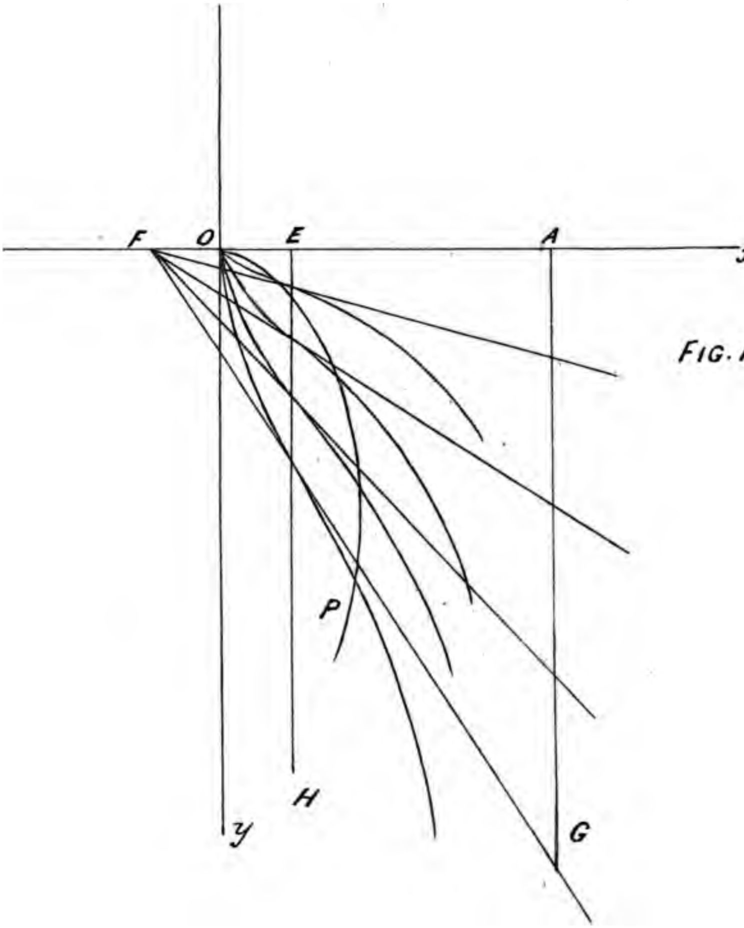


FIG. 19

FIG. 20.

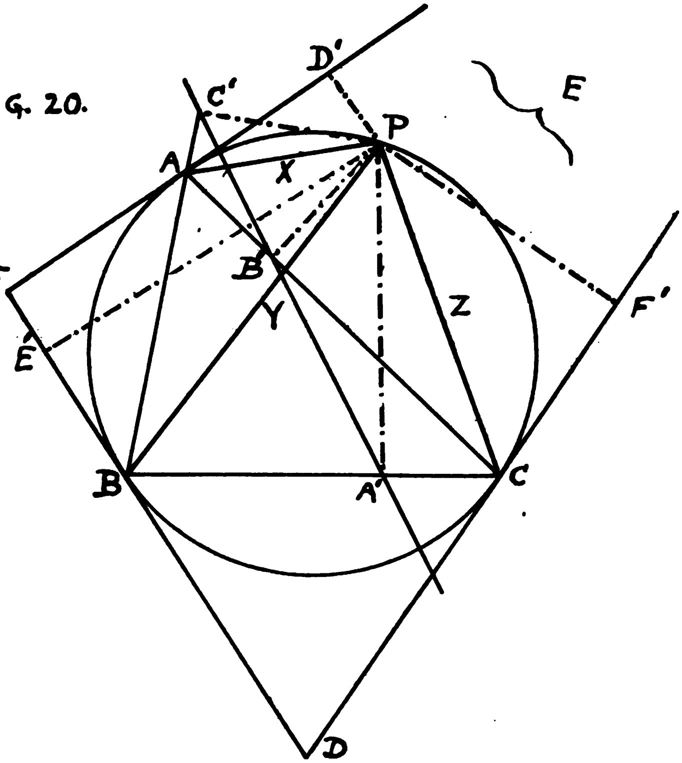


FIG. 21.

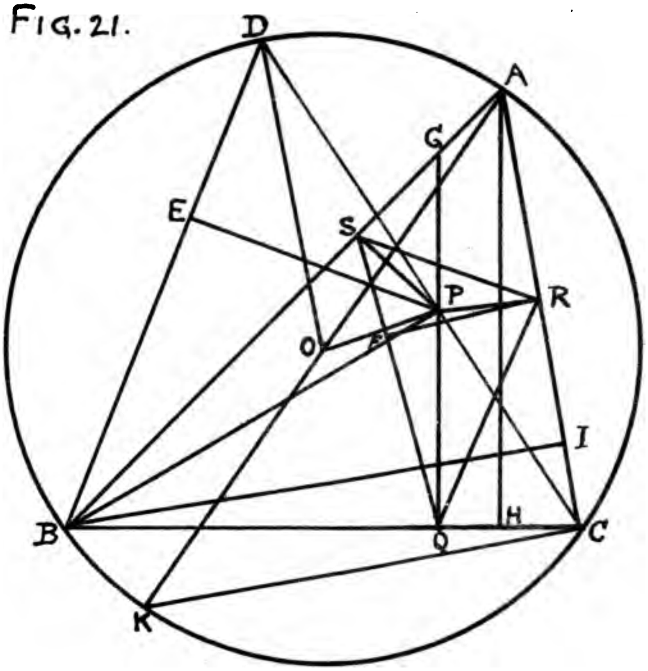


FIG. 22.

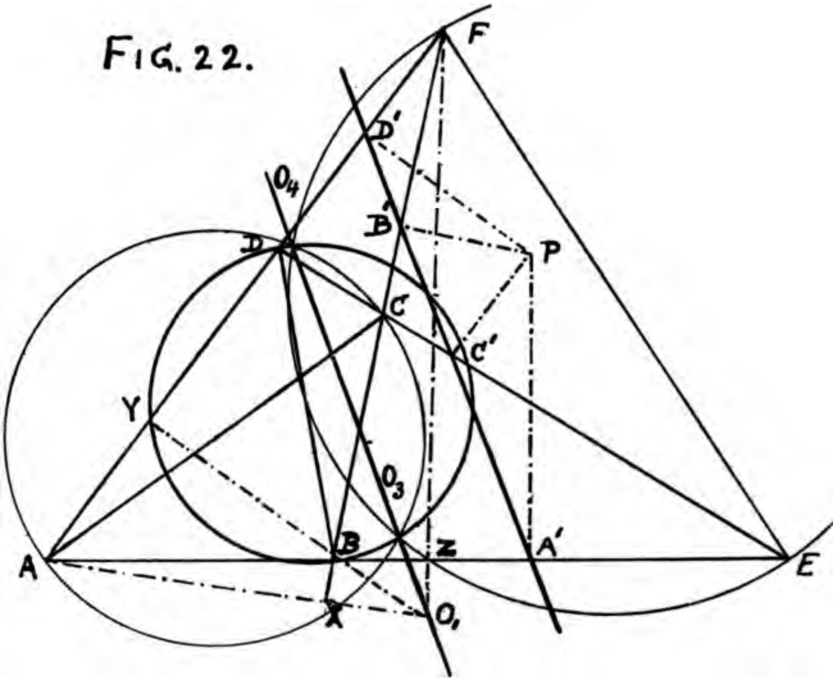

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FIG. 23.

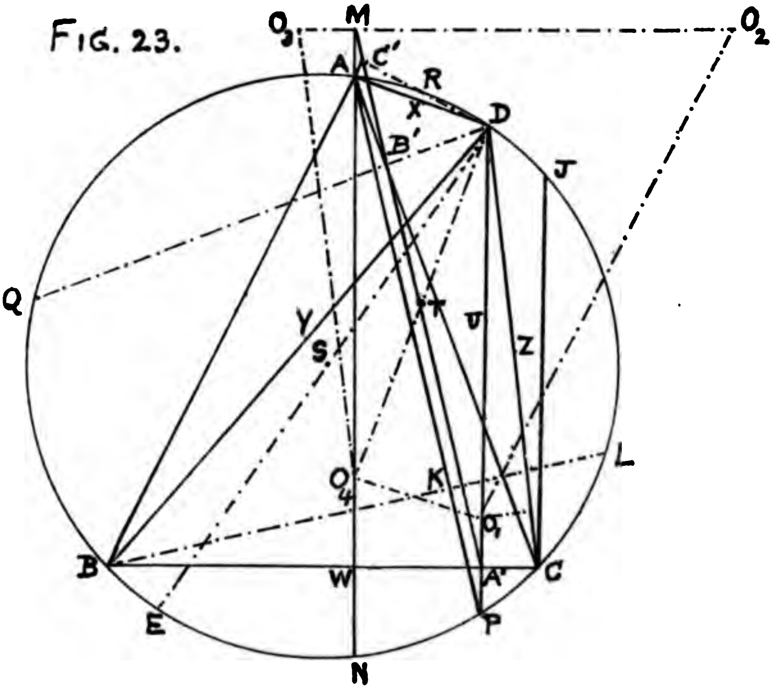
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FIG. 24.

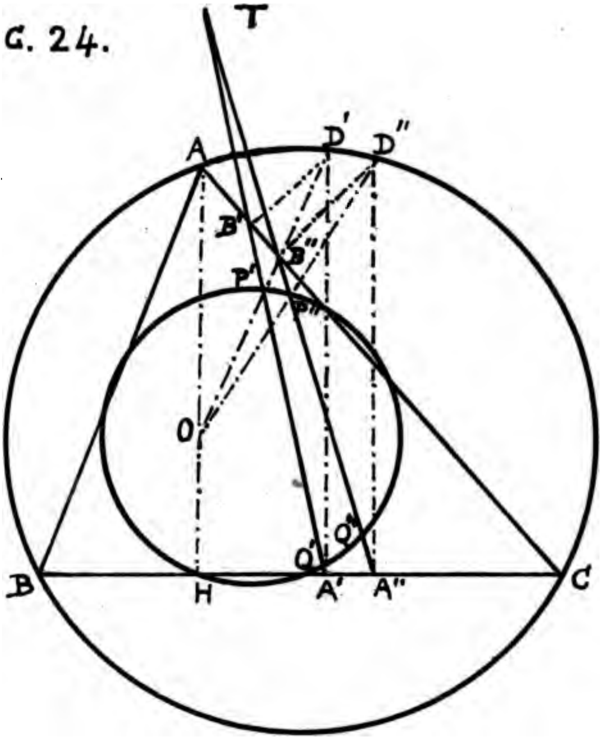


FIG. 25.

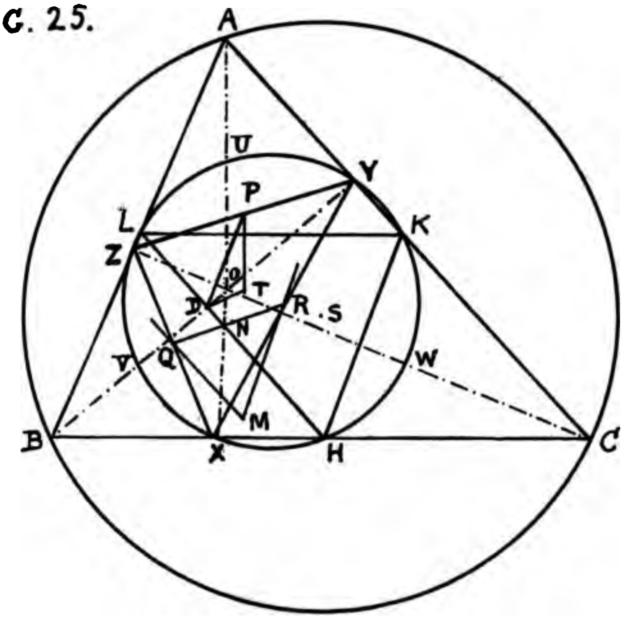


FIG. 26.

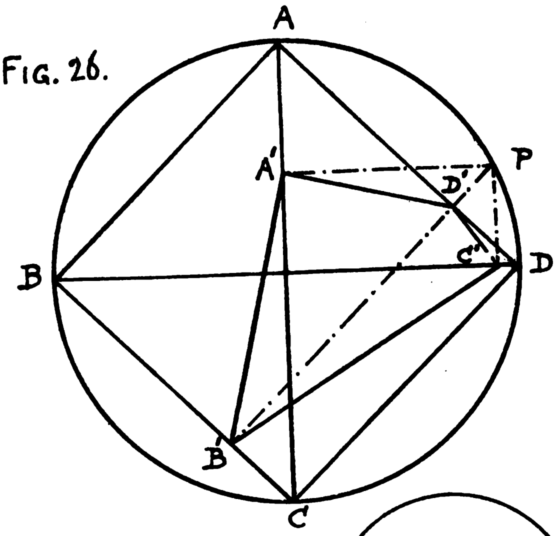


FIG. 27.

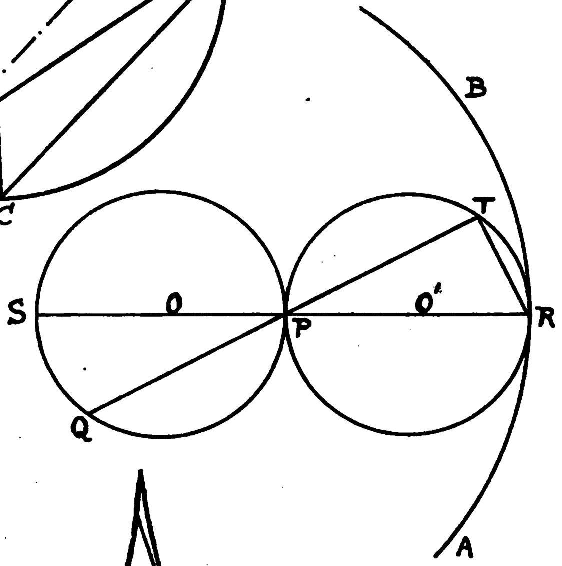


FIG. 28.

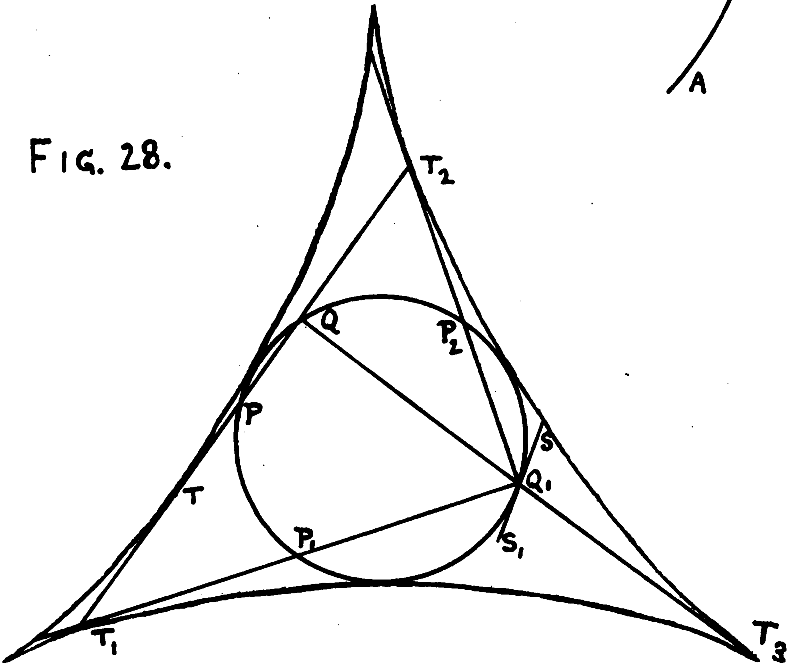
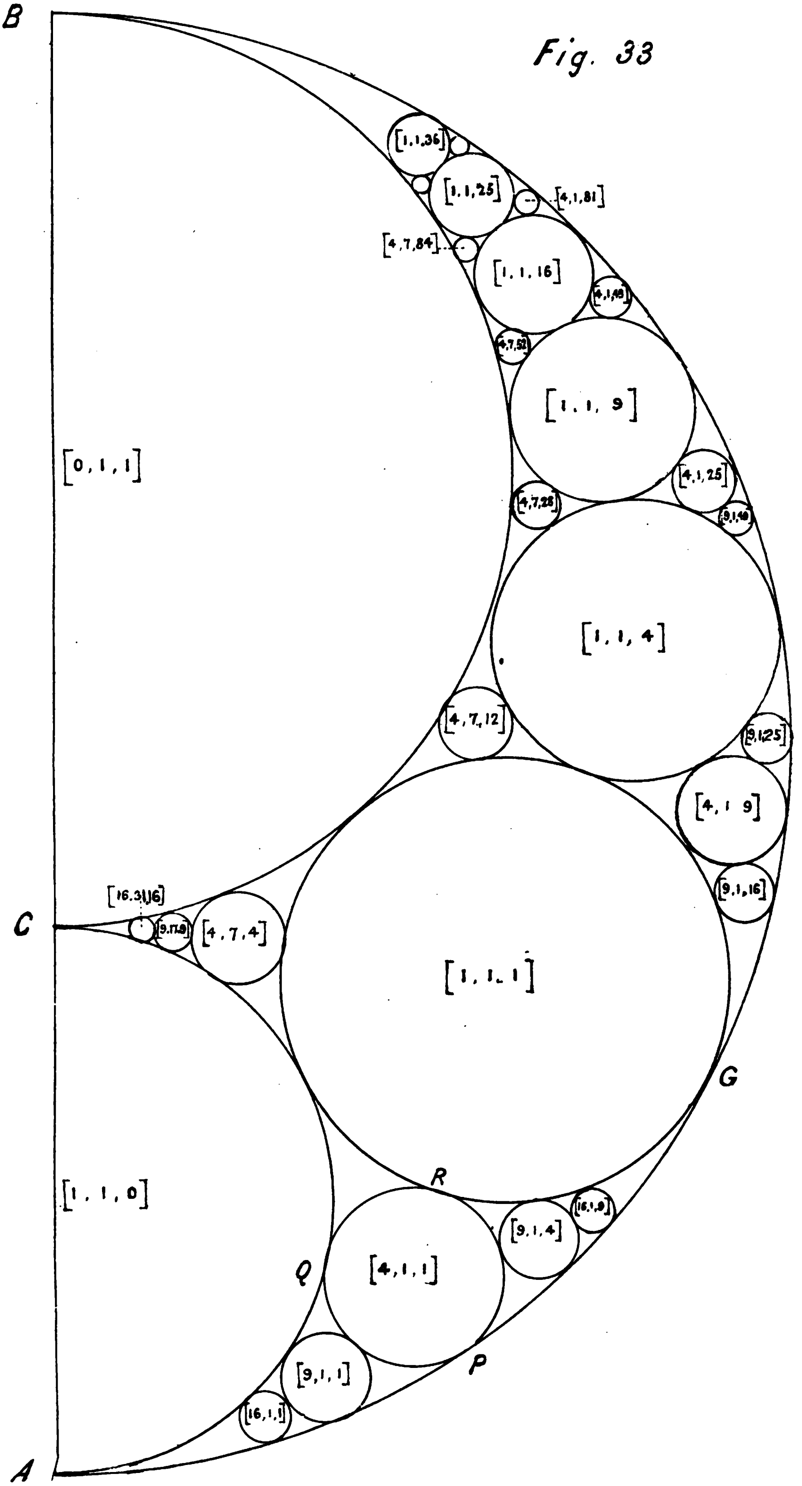


Fig. 33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[16.31,16]
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[9,1,4]
[1,1,0]
Q
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[16,1,1]
A

# PROCEEDINGS

OF THE

# EDINBURGH
# MATHEMATICAL SOCIETY.

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## VOLUME IV.

SESSION 1885-86.

EDINBURGH:
1886.

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VOLUME IV.

SESSION 1885-86.

EDINBURGH:
1886.

PRINTED BY

JOHN LINDSAY, HIGH STREET, EDINBURGH.

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# PROCEEDINGS

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## FOURTH SESSION, 1885-86.

---

### *First Meeting, November 13th*, 1885.

GEORGE THOM, Esq., M.A., Vice-President, in the Chair.

For this Session the following Office-Bearers were elected:—

*President*—Dr R. M. FERGUSON, F.R.S.E.

*Vice-President*—Mr GEORGE THOM, M.A.

*Secretary*—Mr A. Y. FRASER, M.A., F.R.S.E.

*Treasurer*—Mr JOHN ALISON, M.A.

*Committee*—

Messrs R. E. ALLARDICE, M.A.; A. J. G. BARCLAY, M.A.; W. J. MACDONALD, M.A.; J. S. MACKAY, M.A., F.R.S.E.; THOMAS MUIR, M.A., LL.D., F.R.S.E.; and WILLIAM PEDDIE.

## On a theorem in Determinants.

By Sir William Thomson.

---

## The Ancient Methods for the Duplication of the Cube.

By J. S. Mackay, M.A., F.R.S.E.

The object of the present paper is merely to exhibit the methods employed by the ancient Greek geometers in their solution of this celebrated problem. A critical discussion of these methods, of the origin of the problem, and perhaps also an account of more recent researches and a notice of the literature connected therewith may form the subject of a subsequent paper.

The mythical origin of the problem is told by Eratosthenes in his letter to King Ptolemy III. (Euergetes), and mention is there made of the form into which Hippocrates of Chios (about 444 B.C.) recast the problem. Hippocrates's contribution may be set out thus:—

If AB, CD, EF, GH be four straight lines,

then $$AB : GH = \left\{\begin{array}{c} AB : CD \\ CD : EF \\ EF : GH \end{array}\right\}.$$

Now if $AB : CD = CD : EF = EF : GH$,

then $AB : GH =$ triplicate of $AB : CD$,

$= AB^3 : CD^3$.

Hence if $GH = 2\,AB$, $CD^3 = 2\,AB^3$.

If, therefore, $AB^3$ is a given cube, and it is required to double it, take a straight line GH equal to 2AB, and between AB and GH insert two mean proportionals CD, EF. $CD^3$ will be the cube required.

The solutions which follow are translated from the commentary of Eutocius of Ascalon (about 555 A.D.) on Archimedes's treatise *Of the Sphere and Cylinder*. See Torelli's *Archimedis Quae Supersunt Omnia* (Oxonii, 1792), pp. 135-149, or Heiberg's *Archimedis Opera Omnia* (Lipsiae, 1881), pp. 67-127. I have retained the order or disorder in which Eutocius gives the solutions, but have affixed approximate dates to their authors. The solution of Pappus, which is also given by Eutocius, I have translated from Pappus himself. See Commandine's *Pappi Alexandrini Mathematicae Collectiones*, or

Hultsch's *Pappi Alexandrini Collectionis Quae Supersunt* (Berolini, 1876), Book III., proposition 5. The same solution, in almost the same words, occurs again in the Eighth Book of Pappus, and it is from this passage that Eutocius makes his quotation. I have added the remarks which Eutocius makes on Pappus's method. Besides his own solution, Pappus gives those of Eratosthenes, Nicomedes, and Heron, as well as an incorrect solution by a geometer whom he does not name, and his own disproof of the same.

## METHOD OF PLATO (429-348 B.C.).

### Figures 1, 2.

Two straight lines being given to find two mean proportionals in continued proportion.

Let AB, BC at right angles to each other be the two given straight lines between which it is required to find two mean proportionals.

Produce them to D, E.

Construct a rectangular frame FGH, and let a bar KL slide in a groove in one of the legs FG in such a manner as always to be parallel to GH. This will happen if another bar HM be understood to be connected with HG, and to be parallel to FG. For the surfaces FG, HM being grooved from above in dovetail fashion, and pins being fixed in KL to fit these grooves, the movement of KL will always be parallel to GH.

With this construction let any side GH of the angle FGH touch C, and let this angle and the bar KL be shifted until the point G be on the straight line BE, while the side GH touches C; let the bar KL touch the straight line BD at K, and at the other part the point A, so that, as is the case in the figure, the right angle may have a position such as CED, and the bar KL a position such as DA. If this be performed the proposed solution will be effected.

For, since the angles at D and E are right,

$$CB : BE = EB : BD = DB : BA.$$

## METHOD OF HERON (ABOUT 111 B.C.),

## AS GIVEN IN HIS TREATISES ON MECHANICS AND CATAPULTS.

### Figure 3.

Let AB, BC be the two given straight lines between which it is required to find two mean proportionals.

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EDINBURGH:
1886.

With centre B and radius AC describe the circle HKL, and again with centre C and radius AB describe the circle MHN, cutting HKL at H; and join HA, HB, HC.

Then BC is a parallelogram, and HA a diagonal of it. Bisect HA at X, and with centre X describe a circle cutting AB, AC produced at D, E, in such a way however that D and E may be collinear with H. This would happen if a bar cutting AD, AE were moved about H till the lines from X to D, E were equal. If this be done the solution will be obtained.

For the construction is the same as that of Heron and Philon; and it is obvious that the same demonstration will be applicable.

## METHOD OF DIOCLES (ABOUT 180 B.C.?),
### AS GIVEN IN HIS TREATISE ON FIRE-INSTRUMENTS.

Figures 6, 7.

In a circle let two diameters AB, CD be drawn at right angles, and on opposite sides of B let two equal arcs EB, BF be cut off; and through F let FG be drawn parallel to AB, and let DE be joined. I say that FG, GD are two mean proportionals between CG, GH.

Through E draw EK parallel to AB.

Then $EK = FG$, and $KC = GD$.

This will be evident if LE, LF be joined.

For angle $CLE =$ angle FLD, and the angles at K, G are right; therefore, since $LE = LF$, all the parts of the one are equal to all the [corresponding] parts of the other;

therefore the remainder $CK = GD$.

Hence since $DK : KE = DG : GH$,

and $DK : KE = EK : KC$,

for EK is a mean proportional between DK, KC;

therefore $DK : KE = EK : KC = DG : GH$.

Now $DK = CG$, $KE = FG$, $KC = GD$;

therefore $CG : GF = FG : GD = DG : GH$.

If therefore on both sides of B there be taken equal arcs MB, BN, and through N there be drawn NX parallel to AB, and DM be joined, NX, XD will be mean proportionals between CX, XO.

If therefore we draw several consecutive parallels between B and D, and from B towards C set off arcs equal to those cut off by the parallels towards B, and draw straight lines from D to the points thus obtained, like DE, DM, the parallels between B and D will be cut at certain points (O and H in the accompanying figure 6). If these points by the application of a ruler be joined by straight lines we shall have in the circle a certain line described in which if any point be assumed and through it a parallel be drawn to LB, the parallel so drawn and the segment of the diameter towards D will be mean proportionals between the other segment of the diameter towards C, and that part of the parallel between the assumed point and the diameter.

These preliminaries being arranged, let A, B be the two given straight lines between which two mean proportionals are to be found. In the circle whose two diameters CD, EF are at right angles, let there be described by continuous points, as already shown, the curve DHF. As A is to B, so let CG be to GK; join CK and produce it to meet the curve at H; through H draw LM parallel to EF. Then, by what has been already said, ML, LD are the mean proportionals between CL, LH.

Now since $CL : LH = CG : GK$,
and $CG : GK = A : B$;
if between A and B we insert means N, X in the same ratio as CL, LM, LD, LH, then N, X will be the mean proportionals between A and B, as was to be found.

### METHOD OF PAPPUS (ABOUT 300 A.D. ?).

### Figure 8.

A cube is found not only double of a cube by means of the instrument described below and invented by us, but also generally having any prescribed ratio to it.

Let a semicircle ABC be constructed, and from the centre D let BD be drawn at right angles. Let a rule move round the point A in such a manner that the one end of it may be fastened with a nail at A, and the remaining part revolve between B and C round the nail as a centre. This construction being made, let it be enjoined to find two cubes having to each other a given ratio.

Let the ratio of BD to DE be made the same as the given ratio; join CE and produce it to F. Let the rule be moved between B and C till the part of it intercepted between the straight lines FE, EB be equal to the part between the straight line BE and the circumference BKC; for this by continued trials and transference of the rule we

shall easily effect. Let it be done, and let the rule have the position AGHK, so that GH, HK are equal:
I say that the cube described on BD has to the cube described on DH the prescribed ratio, namely that of DB to DE.

Let the circle be understood to be completed;
and having joined KD, produce it to L.
Join LG, which is parallel to BD, because $KH = HG$ and $KD = DL$;
join also AL and LC.

Since the angle GAL in a semicircle is right, and AM is a perpendicular

$$LM : MA = MA : MG;$$

therefore $LM^2 : MA^2 = AM^2 : MG^2$;
and therefore $CM : MA = AM^2 : MG^2$.
With each of these ratios let the ratio of AM to MG be compounded;
then $\left\{\begin{matrix}CM : MA \\ AM : MG\end{matrix}\right\} = \left\{\begin{matrix}AM^2 : MG^2 \\ AM : MG\end{matrix}\right\}$;
that is $CM : MG = AM^3 : MG^3$.
But $CM : MG = CD : DE = BD : DE$,
and $AM : MG = AD : DH = DB : DH$;
therefore $BD : DE$ (the given ratio) $= BD^3 : DH^3$.

Accordingly if, as BD is to DH, we make DH to another straight line such as DN, then DH, DN will be two mean proportionals between BD, DE.

Now, it is to be observed that this construction is the same as that which Diocles has proposed, and differs only in this that he describes by consecutive points between A and B a certain curve, on which the point G is taken by producing CE to cut the said curve, while here the point G is found by means of a rule movable round A. That the point G is the same whether it is obtained as here, by means of a rule, or as Diocles says, we may learn thus.

Produce MG to N,* and join KN [cutting DB at X].
Since $KH = HG$, and GN is parallel to HB,
therefore $KX = XN$.
Now XB is common and at right angles [to KN],

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* This addition to Pappus's figure is left to be made by the reader. The point N here is different from the point N spoken of in the last sentence of Pappus's solution.

for KN is bisected perpendicularly by the line through the centre; therefore the base is equal to the base, and, on that account, the circumference KB is equal to BN.

The point G, therefore, is the point on Diocles's curve, and the demonstration is the same.

For Diocles said that

$$CM : MN = MN : MA = AM : MG;$$

and $NM = ML$, for the diameter cuts it perpendicularly;

therefore $CM : ML = LM : MA = AM : MG$.

Hence between CM, MG, the mean proportionals are LM, MA.

But $CM : MG = CD : DE$,

and $CM : ML = AM : MG = CD : DH$;

therefore when two mean proportionals between CD, DE [are to be found], the second [line, that is the first mean proportional] is DH, which also Pappus obtained.

## Method of Sporus.*

### Figure 9.

Let AB, BC be the two given unequal straight lines; it is required to find between AB, BC two mean proportionals in continued proportion.

From B draw DBE at right angles to AB;

with centre B and radius BA describe the semicircle DAE.

Join EC, and produce it to F; and from D let a straight line be drawn so that GH is equal to HK; for this is possible.

From G and K let the perpendiculars GL, KNM be drawn to DE.

Since $KH : HG = MB : BL$,

and $KH = HG$; therefore $MB = BL$;

therefore the remainder $ME = LD$;

therefore the whole $DM = LE$;

therefore $MD : DL = KM : GL$.

Again since $DM : MK = KM : ME$;

therefore

$$\begin{aligned} DM : ME &= DM^2 : MK^2 \\ &= DB^2 : BH^2 \\ &= AB^2 : BH^2, \text{ for } DB = AB. \end{aligned}$$

Again since $MD : DB = LE : EB$;

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* Eutocius is, as far as I know, the only ancient author who mentions this geometer.

but $MD : DB = KM : HB$,
and $LE : EB = GL : CB$;
therefore $KM : HB = GL : CB$.
Now $KM : GL = MD : DL$,
$= DM : ME$,
$= AB^2 : HB^2$;
therefore $AB^2 : HB^2 = BH : BC$.

Between HB, BC take a mean proportional X.

Since $AB^2 : BH^2 = HB : BC$;
but $AB^2 : BH^2 =$ duplicate of $AB : BH$,
and $HB : BC =$ duplicate of $HB : X$;
therefore $AB : BH = BH : X$.
But $BH : X = X : BC$;
therefore $AB : BH = BH : X = X : BC$.

It is obvious that this method is the same as that of Pappus and Diocles.

## METHOD OF MENECHMUS (ABOUT 350 B.C.).

Figure 10.

Let A, E be the two given straight lines; it is required to find two mean proportionals between A, E.

Let it be done, and let them be B, C;
and let the straight line DG, given in position and terminated at D, be drawn.
At D let DF be made equal to C, and let HF be drawn at right angles, and FH made equal to B.

Since then the three straight lines A, B, C are proportional, the rectangle $A \cdot C = B^2$;
therefore the rectangle under the given line A and C, that is DF, $= B^2 = FH^2$;
therefore H lies on a parabola described through D.

Let the parallels HK, DK be drawn.

Since the rectangle $B \cdot C$ is given, for it is equal to the rectangle $A \cdot E$;
therefore the rectangle $KH \cdot HF$ is also given;
therefore the point H lies on a hyperbola with KD, DF for asymptotes.
The point H is therefore given, and therefore also the point F.

The synthesis will be the following:—

Let A, E be the given straight lines, and let DG be given in position and terminated at D. Through D let there be described a parabola whose axis is DG and parameter [literally, right side of the figure] A. Let the squares of the ordinates drawn at right angles to DG be equal to the rectangles [literally, let the straight lines drawn at right angles to DG be equal in power to the spaces] applied to A, and having as breadths the lines cut off by them to the point D. Let DH be the parabola, and let DK be perpendicular.
With KD, DF as asymptotes let a hyperbola be described such that the lines drawn parallel to KD, DF shall make a space equal to the rectangle $A \cdot E$; it will cut the parabola. Let it cut it at H, and let the perpendiculars HK, HF be drawn.

Since then $FH^2 = A \cdot DF$;
therefore $A : FH = HF : FD$.
Again since $A \cdot E = HF \cdot FD$;
therefore $A : FH = FD : E$.
But $A : FH = FH : FD$;
therefore $A : FH = FH : FD = FD : E$.
Let B be made $= HF$, and $C = DF$;
then $A : B = B : C = C : E$;
therefore A, B, C, E are in continued proportion, as was to be found.

### OTHERWISE.

Figure 11.

Let the two given straight lines AB, BC be at right angles to each other, and let the means between them be DB, BE, so that

$$CB : BD = BD : BE = BE : BA,$$

and let DF, EF be drawn at right angles.

Since $CB : BD = DB : BE$;
therefore the rectangle $CB \cdot BE$, that is the rectangle under a given line and $BE = BD^2 = EF^2$.
Since then the rectangle under a given line and $BE = EF^2$; therefore F touches a parabola about the axis BE.

Again since $AB : BE = BE : BD$;
therefore the rectangle $AB \cdot BD$, that is the rectangle under a given line and $BD = EB^2 = DF^2$;
therefore F touches a parabola about the axis BD.
But it touches also another given [parabola] about [the axis] BE; therefore F is given.

And FD, FE are perpendiculars;
therefore D, E are given.

The synthesis will be the following:—

Let the two given straight lines AB, BC be at right angles to each other, and let them be produced indefinitely from B. About the axis BE let there be described a parabola whose parameter is BC. Again about the axis DB let there be described a parabola whose parameter is AB. These parabolas will cut each other; let them cut at F, and from F let the perpendiculars FD, FE be drawn.

Since from the parabola FE, that is DB, has been drawn,
therefore $CB \cdot BE = BD^2$;
therefore $CB : BD = DB : BE$.
Again since from the parabola FD, that is EB, has been drawn,
therefore $DB \cdot BA = EB^2$;
therefore $DB : BE = BE : BA$.
But $DB : BE = CB : BD$;
therefore $CB : BD = BD : BE = EB : BA$,
as was to be found.

The parabola is described by means of the compass invented by the engineer Isidorus of Miletus, our master, and described by him in the commentary he wrote on Heron's *Arches*.

### METHOD OF ARCHYTAS (ABOUT 400 B.C.), AS EUDEMUS GIVES IT.

Figure 12.

Let the two given straight lines be AD, C: it is required to find two mean proportionals between AD, C.

About the greater AD describe a circle ABDF, and in it let there be inflected AB equal to C; let AB produced meet at P the tangent to the circle drawn from D, and let BEF be drawn parallel to PD. Let a semicylinder be conceived to be situated uprightly on the semicircle ABD, and on AD an upright semicircle lying in the parallelogram of the cylinder. Now, while A the end of the diameter remains fixed, if this semicircle be revolved from D towards B, it will in its revolution cut the cylindrical surface, and will describe on it a certain line. Again, if, while AD remains fixed, the triangle APD be revolved with a motion opposite to that of the semicircle, it will describe a conical surface with the straight line AP, and this in its revolution will meet the line on the cylinder in some point; at the same time B will describe a semicircle on the surface of the cone. At the place where these lines meet, let the revolving semicircle have the position DKA, and the oppositely revolving triangle the position DLA; and let K be the point of the said meeting. Let also the semicircle described by the point B be BMF, and let the common section of it and the circle BDFA be BF. From K to the plane of the semicircle BDA let a perpendicular be drawn: it will fall on the circumference of the circle, because the cylinder stands uprightly. Let it be KI, and let the line joining I to A meet BF at H, and the line AL meet the semicircle BMF at M. Let KD, MI, MH be joined.

Then since both the semicircles DKA, BMF are perpendicular to the underlying plane, their common section MH is perpendicular to the plane of the circle, so that MH is perpendicular to BF;
therefore $HB \cdot HF$, that is $HA \cdot HI = MH^2$.
Hence triangle AMI is similar to both triangles MIH, MAH, and angle IMA is right.
Now angle DKA is right;
therefore KD, MI are parallel, and from the similarity of the triangles there will be proportional

$$DA : AK = KA : AI = IA : AM;$$

therefore the four DA, AK, AI, AM are continually proportional.
Now AM is equal to C, since it is equal to AB;
therefore between the two given lines AD, C two mean proportionals have been found AK, AI.

## METHOD OF ERATOSTHENES (276–194 B.C.).

To King Ptolemy Eratosthenes greeting.

They say that an ancient tragedian introduced Minos constructing a tomb for Glaucus, and that when he learned that it was to be a hundred feet every way, he said: You have indicated a small enclosure for a royal tomb: let it be double; and failing not of the beautiful form, double quickly each side of the tomb. But it appeared that he had made a mistake; for when the sides were doubled, the surface became quadrupled, and the solidity octupled. Now, the inquiry was made by geometers how one could double a given solid and retain it in the same shape, and such a problem was called the duplication of the cube; for having supposed a cube they sought to double it. When all were for a long time at a loss, Hippocrates of

Chios was the first to perceive that if between two straight lines, the greater of which was double the less, there could be found two mean proportionals in continued proportion, the cube would be doubled, so that his difficulty was exchanged for another difficulty equally great. At a later time, it is said that certain Delians, in obedience to an oracle, attempted to double one of the altars, and fell into the same difficulty. They sent accordingly to the geometers, who were with Plato at the Academy, requesting them for a solution of the problem. While they were industriously applying themselves to the matter, and endeavouring to find two means between the two given lines, Archytas of Tarentum is said to have discovered them by means of his semi-cylinders, and Eudoxus by means of the so-called curved lines. Now, all these succeeded in writing demonstratively, but they were not able to make practically useful applications of their methods, except Menechmus to some small extent, and that with difficulty. But an easy instrumental method has been contrived by us whereby between the two given lines we shall find not only two means, but as many as any one shall prescribe. This being found, we shall be able to change into a cube any given solid contained by parallelograms, or to transform from one shape into another, and to make similar, and to increase while preserving the similarity, as, for example, altars and temples. We shall be able also to change into a cube liquid and dry measures, such as a bushel or a medimnus, and by means of its side measure the capacity of these vessels. The contrivance will be useful also to those who wish to enlarge instruments for throwing bolts or stones. For all the parts must be enlarged proportionally, thicknesses, lengths, apertures, naves (?), and inserted cords, if the discharge is to be proportionally increased. Now, these things cannot be performed without finding the means. The demonstration and the construction of the said instrument I have explained to you below.

Figures 13, 14.

Let AE, DH be the two given unequal straight lines between which it is required to find two mean proportionals in continued proportion.

On any straight line EH let AE be placed perpendicularly, and on EH let three consecutive [equal] parallelograms be constructed AF, FI, IH, and in them let there be drawn the diagonals AF, LG, IH which will therefore be parallel. Let the middle parallelogram FI remain fixed, but let the parallelogram AF be pushed above the middle one, and IH below it, as in the second diagram [fig. 14] till the points A, B, C, D are in the same straight line. Through the points A, B, C, D let a straight line be drawn, and let it meet EH produced at K.

Then from the parallels AE, FB,

$$AK : KB = EK : KF;$$

and from the parallels AF, BG,

$$AK : KB = FK : KG;$$

therefore $AK : KB = EK : KF = KF : KG.$

Again from the parallels BF, CG,

$$BK : KC = FK : KG;$$

and from the parallels BG, CH,

$$BK : KC = GK : KH;$$

therefore $BK : KC = FK : KG = GK : KH.$

But $FK : KG = EK : KF;$

therefore $EK : KF = FK : KG = GK : KH.$

Now $EK : KF = AE : BF,$

$FK : KG = BF : CG,$

and $GK : KH = CG : DH;$

therefore $AE : BF = BF : CG = CG : DH.$

Hence between AE, DH two means BF, CG have been found.

These things then have been demonstrated on geometrical surfaces. But in order that we may be able to find the two means instrumentally, let there be constructed a rectangular frame of wood, ivory, or bronze, having three equal tablets as thin as possible, of which the middle one is fixed, but the two others can be moved along in grooves of such a size and fit as anybody may please; for what relates to the demonstration will be carried out in the same way. But in order that the lines may be found more accurately, care must be taken in moving the tablets that all [corresponding] lines may remain parallel, and the parts fit each other smoothly and without gaps. Now, in the votive offering the instrument is of bronze, and is fastened with lead just under the crown of the slab; under it is the demonstration concisely worded and the diagram, and finally the epigram. Let me therefore transcribe these things for you, that you may have them exactly as they are on the votive offering. Of the two diagrams, only the second has been drawn on the slab.

Between two given straight lines to find two mean proportionals in continued proportion.

Let AE, DH be given.

In the instrument I move the tablets till the points A, B, C, D, are in the same straight line, as in the diagram, suppose.

Then AK : KB = EK : KF, from the parallels AE, BF;
and = FK : KG, from the parallels AF, BG;
therefore EK : KF = KF : KG.
Now these same are to each other as AE : BF and BF : CG.
Similarly we shall show that
FB : CG = CG : DH.
therefore AE, BF, CG, DH are proportional.
Between the two given straight lines therefore two means have been found.

But if the given lines should not be equal to AE, DH; having made AE, DH proportional to them, we shall take the means between these [that is, AE, DH], and shall transfer them [by proportion] to those, and we shall have performed what was prescribed. Should it be enjoined to find several means, we shall place in the instrument one more tablet than the number of means to be found. The demonstration will be the same.

[The epigram is here omitted.]

## METHOD OF NICOMEDES (ABOUT 180 B.C.), AS GIVEN IN HIS TREATISE ON CONCHOIDAL LINES.

Figure 15.

Let CD, DA perpendicular to each other be the two given straight lines between which it is required to find two means continually proportional.

Complete the parallelogram* ABCD, and bisect the sides AB, BC at the points L, E.
Having joined LD produce it and let it meet CB produced at G; draw EF perpendicular to BC, and make CF equal to AL.
Join FG, and make CH parallel to it;
and since an angle is contained by KC, CH, from F let there be drawn FHK, making HK equal to AL or CF (which has been shown to be possible by the conchoid).
Having joined KD produce it and let it meet BA produced at M:
then CD : KC = KC : MA = MA : AD.

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* The parallelogram is lettered ABCL in Eutocius, ABCD in Pappus. The Greek letters for D and L are easily confounded.

Since BC is bisected at E and KC is added to it,
therefore $BK \cdot KC + CE^2 = EK^2$.
Add $EF^2$ to both sides;
then $BK \cdot KC + CE^2 + EF^2 = KE^2 + EF^2$;
therefore $BK \cdot KC + CF^2 = KF^2$.
And since $MA : AB = MD : DK$,
and $MD : DK = BC : CK$;
therefore $MA : AB = BC : CK$.
Now $AL = \frac{1}{2}AB$, and $CG = 2BC$;
therefore $MA : AL = GC : CK$.
But $GC : CK = FH : HK$, on account of the parallels GF, CH;
therefore $ML : LA = FK : KH$, by composition.
Now $AL = HK$, since $AL = CF$;
therefore $ML = FK$, and therefore $ML^2 = FK^2$.
And $BM \cdot MA + LA^2 = ML^2$,
and $BK \cdot KC + CF^2 = FK^2$,
of which $AL^2 = CF^2$, for $AL = CF$;
therefore the remainder $BM \cdot MA$ = the remainder $BK \cdot KC$;
and therefore $MB : BK = KC : AM$.
But $BM : BK = CD : CK$;
therefore $CD : CK = CK : AM$.
And $DC : CK = MA : AD$;
therefore $DC : CK = CK : AM = AM : AD$.

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## Notes on Euclid I., 47.

By WILLIAM HARVEY, B. A.

ABC is a triangle, right-angled at A, and X, Y, Z are the centres of the squares described on the sides opposite the angles A, B, C; XD, XM, XN are respectively perpendicular to BC, CA, AB; MY, DY are joined, and DY meets AC at E; NZ, DZ are joined, and DZ meets AB at F.

In fig. 16 the squares are drawn outwardly, and the points X, Y, Z are on the sides of BC, CA, AB remote from the opposite angles A, B, C.

In fig. 17 the squares are drawn inwardly, and the points X, Y, Z are on the same sides of BC, CA, AB as the opposite angles A, B, C.

The properties depending on the points X, Y, Z are analogous in these two cases.

Properties.

(1) D is the middle point of BC, and $XD = \frac{1}{2}BC$.

(2) YD is parallel to AB, and
$= \frac{1}{2}(CA + AB)$ (fig. 16.)
$= \frac{1}{2}(CA - AB)$. (fig 17.)

For YD joins the middle points of two sides of a triangle, and is, therefore, parallel to the third side and equal to half of it.

ZD is parallel to AC, and
$= \frac{1}{2}(CA + AB)$. (fig. 16.)
$= \frac{1}{2}(CA - AB)$. (fig. 17.)

For the above proof applies equally to YD and ZD.

(3) Hence $YD = ZD$, and the angle $ZDY =$ a right angle.

(4) E and F are the middle points of CA, AB respectively.

(5) The figure AMXN is a square, whose side
$= \frac{1}{2}(CA + AB)$. (fig. 16.)
$= \frac{1}{2}(CA - AB)$. (fig. 17.)

This follows from the congruency of the triangles CMX, BNX.*

(6) DY is equal and parallel to XM, and DXMY is a parallelogram, whose perimeter
$= \frac{1}{2}(CA + AB + BC)$. (fig. 16.)
$= \frac{1}{2}(CA - AB + BC)$. (fig. 17.)

This follows from (2) and (5).
Hence $YM = XD = \frac{1}{2}BC$.

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* My attention was drawn, after the present paper was in proof, to an anticipation of the above property in the *Educational Times*, for August 1879, (Vol. 32, p. 241), where the mathematical editor, Mr J. C. Miller, proposes and solves the following proposition:

A square is constructed on the hypotenuse of a right-angled triangle; prove that the distance of the middle of this square from each of the sides that contain the right angle of the triangle is equal to half the *difference*, or to half the *sum*, of these sides, according as the triangle and the square are on the *same* or on *opposite* sides of the hypotenuse.

DZ is equal and parallel to XN, and DXNZ is a parallelogram whose perimeter is equal to that above, and

$$ZN = XD = \tfrac{1}{2}BC.$$

(7) ZN is equal and parallel to YM, and ZNMY is a parallelogram.
Therefore ZY = NM.

(8) The points D and A are similarly situated on MN and ZY, and the triangles AMN, DZY are congruent.
For AM is equal and parallel to DZ, by (3) and (5); therefore DM is equal and parallel to ZA.

Similarly DN is equal and parallel to YA.
Hence D lies on the line MN.

(9) AX bisects the angle BAC. (fig. 16.)
AX bisects the supplement of the angle BAC. (fig. 17.)

(10) AX is equal and perpendicular to YZ,
BY is equal and perpendicular to ZX,
CZ is equal and perpendicular to XY.
For AX = MN = ZY;
and AX bisects the angle BAC or the supplement of angle BAC, by (5) and (7),
while ZY bisects the supplement of angle BAC or the angle BAC; therefore AX is perpendicular to ZY.

Again, from the congruency of the triangles BZY, ZAX, it follows that BY is equal to ZX; and since one of the sides ZY of the first triangle is perpendicular to the corresponding side AX of the second, each side of the first is perpendicular to the corresponding side of the second; and therefore BY is perpendicular to ZX.

The same is true, by parity of conditions, for CZ and XY.

(11) AX, BY, CZ are respectively perpendicular to the sides of the triangle XYZ; therefore they meet in a point O which is the orthocentre of the triangle XYZ.
This follows from (10).

(12) The area of the triangle XYZ is equal to that of the square AMXN.

For the triangle XYZ is on the same base YZ as the triangle DYZ, and the altitude of the former is twice that of the latter, since XM is equal and parallel to DY. Therefore the area of XYZ is twice that of DYZ, and DYZ is equal to AMN, by (8), which is half the square AMXN. Hence the triangle XYZ is equal to the square AMXN.

(13) $\triangle BCX + \triangle ABC =$ square AMXN. (fig. 16.)
$\triangle BCX - \triangle ABC =$ square AMXN. (fig. 17.)
This follows from the congruency of the triangles BNX, CMX.*

(14) $\triangle AYC + \triangle AZB + \triangle ABC =$ square AMXN. (fig. 16.)
$\triangle AYC + \triangle AZB - \triangle ABC =$ square AMXN. (fig. 17.)
For in the first case the sum of the areas is equal to the quadrilateral BCYZ, which, since D is the middle point of BC, is equal to twice the triangle DYZ, that is, to the square AMXN.

In the second case, the expression on the left side is equal to $\triangle BCY - \triangle BZY$, and $\triangle BCY = \triangle CYZ$, since BZ is parallel to CY. And since D is the middle point of BC, and B and C are on opposite sides of the base ZY

$$\triangle CYZ - \triangle BYZ = 2\triangle DYZ = \text{square AMXN}.$$

(15) In either case we prove by (13) and (14) that

$$\triangle BCX = \triangle ABZ + \triangle ACY$$

which gives the result of the 47th proposition.

(16) There are eight varieties of Euclid's figure which result from placing the squares in different positions with respect to the sides of the triangle. In the two we have considered the properties of the points X, Y, Z are the most interesting, and are strictly analogous. The properties furnished by the remaining cases are, with a few exceptions, not analogous to the above, but are of some interest in themselves.

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* From this we get a geometrical proof of the result $\Delta = s(s-a) = (s-b)(s-c)$ for the area of a right-angled triangle.

For, with the usual notation, the result of (13) becomes

$$\Delta + \tfrac{1}{4}a^2 = \tfrac{1}{4}(b+c)^2 \quad \text{(fig. 16).}$$
$$-\Delta + \tfrac{1}{4}a^2 = \tfrac{1}{4}(b-c)^2 \quad \text{(fig. 17).}$$

which give the two expressions for the area.

# On Physical Science in Schools.

By A. J. G. Barclay, M.A.

[*Abstract.*]

Now that more science has become the popular educational cry, there is a danger of raising too great expectations of what physical science can do, and so of paving the way for a reaction against it when it is found not to yield the results unduly expected of it. This arises mainly from basing the claims of physical science to a school place upon an exaggerated estimate of the value of the knowledge imparted, and from not admitting it as an educative agent capable of filling a unique place in the educational course. On account of the popular belief in the ultimate practical, or bread-and-butter value of science teaching, it has been introduced in many cases as an attraction in a school prospectus without adequate means being provided for efficiently carrying it on. In such cases it often takes the form of the popular lecture illustrated by experiments which requires no great mental application on the pupil's part, gives him amusement, and relieves him for the time from some of his dry daily routine, but which fills him at the same time with a false and mischievous notion of what science is. It is of great importance that the true place and aim of physical science in schools should be clearly recognised, not merely by scientific men and educationists, but also by the intelligent general public, for it is only when such recognition is general that the means of equipping and maintaining science work will be forthcoming.

The Science and Art Department by means of its grants in aid has done a great deal to introduce and extend science teaching, and School Boards find the temptation of earning grants so strong that they readily equip their common schools with the needful apparatus and staff for this work, while they leave their secondary schools most scantily provided in both these respects, the science work usually being thrown on the shoulders of the already well-loaded mathematical masters. While we all acknowledge the evils of any teaching undertaken for examination purposes or for earning grants, yet such science teaching has done good, and though not the best, it is better to have it than to have no science instruction at all.

Whenever science teaching is lifted out of the degrading position of being a mere attraction in a school prospectus, and is acknowledged as an important factor in true education with a special bearing on the problems of life, then will due provision be made for it in suitable rooms properly furnished and supplied with apparatus. Teachers will be engaged with special reference to this work, and proper time will be allowed them for preparation, with qualified assistance where needed. Then also, and this is of considerable importance, the progress of the pupils in this department will be duly recognised, and take a place in the honours of the school.

"When scientific physics have as recognised a place in public instruction as Latin and Greek they will be as well taught." (M. Arnold).

The first physical science subject requiring a lift out of its present position is physical geography or, more generally, physiography. This subject, the first branch of physical science the pupil is introduced to, is commonly left to the English teacher to be taken up or not as he can find time for it amid the multiplicity of things he has to teach, and in that case receives very inadequate treatment. It ought, instead of being hustled in among a crowd of other claimants on the teacher's time, to be specially provided for in the time-table under a science master, and furnished with suitable appliances, especially physical maps and diagrams, the best procurable.

Chemistry is taught usually both theoretically and practically, and physics theoretically only. Practical physics is either not attempted at all, or as a rule not well done. Physical laboratory work is looked upon as unsuitable, taking the form of "qualitative experiments only leading to play," or of "measurements involving the use of costly instruments and taking too long time." In an excellent book on practical physics, by A M. Worthington, it is shown how without any very costly instruments or furnishings a most interesting and useful course of experiments can be given to classes of even young boys. It accustoms them to the use of their hands, and to take observations with common instruments, giving them a training in exact methods of the highest value. It is a kind of training most suitable for boys, a training which, when left to a later period, disgusts many students of University physical laboratories, who cannot be entrusted with higher work for the want of such a preliminary training. The extension of such classes is greatly to be desired for the sake of pupils who may be looking forward to any kind of physical work, and for the sake of relieving the University laboratories of a great amount of the elementary work which is their reproach.

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*Second Meeting, December 11th*, 1885.

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Dr Ferguson, F.R.S.E., President, in the Chair.

## On certain Integrals.

### By Professor Tait.

This paper was based mainly on the results of an investigation which will appear in full in the *Transactions* of the Royal Society of Edinburgh. Incidentally, however, it led to a discussion of the question:—*Find the law of density of a planet's atmosphere, supposing Boyle's law to be true for all pressures, and the temperature to be uniform throughout.*

Boyle's law gives $p = k\rho$, where $\rho$ is the density at distance $r$ from the planet's centre.

The Hydrostatic condition is $\frac{dp}{dr} = -\rho\mathrm{R}$, where R is the attraction on unit of mass.

Hence $k\frac{d\rho}{dr} = -\rho\frac{\mathrm{M} + \int_{r_0}^{r} 4\pi r^2\rho dr}{r^2}$, where $r_0$ is the radius, and M the mass of the planet.

Write this as

$$\frac{kr^2}{\rho}\frac{d\rho}{dr} = -\mathrm{M} - \int_{r_0}^{r} 4\pi r^2\rho dr$$

and differentiate; and we obtain the curious equation

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{d\rho}{dr}\right) = -\frac{4\pi}{k}r^2\rho \quad \ldots \quad \ldots \quad (1)$$

A *special* value of $\rho$ (corresponding to the absence of a nucleus) is

$$\rho = +\frac{k}{2\pi r^2},$$

but this cannot be generalised.

The finding of the integral of (1) in a form convergent for all values of $r$ greater than $r_0$ presents novel and grave difficulties; but it is clear from the physical question on which the whole is based that such a solution exists.

If we change the independent variable to $s$, where $rs = 1$, (1) becomes

$$\frac{d^2 \log \rho}{ds^2} = -\frac{4\pi}{k}\frac{\rho}{s^4};$$

or, if $\log \rho = u$, $\quad \frac{4\pi}{k} = e$,

$$\frac{d^2 u}{ds^2} = -\frac{e}{s^4}\epsilon^u.$$

This seems to be the simplest form into which the equation can be transformed.

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## To transform a rectangle into a square.

By W. Peddie.

Let $l$ be the length of the rectangle and $b$ its breadth, while $s$ is the side of the square equal to it in area; $s$ is found, of course, by taking $l : s = s : b$. A number of different cases arise.

I. $2s < l + b$. Fig. 18 represents the method of cutting the rectangle in this case. AC is the rectangle and AF the equal square. Make HM = BC, QF = MC, NB = QM. The proof is evident.

II. $2s = l + b$. In this case EB = QF, BC = DQ, ∴ EC is similar and equal to GQ.

III. $2s > l + b$. Three cases arise.

- $\alpha$. $2s > l$. This includes method I., and the method shown in Fig. 19.
- $\beta$. $2s = l$. Then AQ = EC.
- $\gamma$. $2s < l$.

Let $l = (k + \theta)s$, where $k$ is an integer and $\theta$ a proper fraction. If we cut off portions of the rectangle of length $s$, the above method applies directly if $2\theta s \nless s$ *i.e.*, if $\theta \nless \frac{1}{2}$. Again if $\theta < \frac{1}{2} = \frac{1}{p}$ where $p$ is an integer, the solution is easy. If $p$ is an improper fraction consistent with the condition $\theta < \frac{1}{2}$, the simplest method is to cut up the rectangle into an equal number of parts, so as to form a rectangle of suitable breadth. In some of these cases the transformation is produced by simple sliding of the various parts parallel to themselves. In others the parts have to be rotated through a right angle.

Another solution of the problem is given in Dr Charles Hutton's *Recreations in Mathematics and Natural Philosophy.*

# The Theory of Contours, and its Applications in Physical Science.

By W. Peddie.

## PART I.

1. The word *contour* is largely used in ordinary language, but its meaning, when so used, is in general very different from its meaning as a scientific term. We speak of the *contour* of a hill, a cloud, a country, and so on; meaning usually a *profile* or an *outline*,—sometimes a particular outline only. Yet, even in this popular use of the word, we have an indication of its more exact significance. Thus, we see that the visible horizon, if we consider it to be a contour line, is the curve in which the earth's surface is met by its tangent-cone the vertex of which is at the observer's eye. The tangent-surface has a constant characteristic; and it is this possession of a distinctive property by all surfaces which give rise to contour lines, which furnishes the reason for the peculiar applicability of the method of contours to physical problems.

2. The word is used in a more special sense by ordnance surveyors. In their terminology a contour line is a line drawn on a map of a country through points which are at a constant height above mean sea-level. Here again we have a quantity which is constant all along the line, and also all over the surface which gives rise to the line by its intersection with the earth. This is a *level* surface, or surface of *equilibrium.* It is approximately true that the special characteristic is constant *height* above sea-level, so long as the area considered is so small that gravity has sensibly the same value all over it. Because of the variation of gravity from point to point of the earth's surface, and because of the earth's rotation, &c., it is not true that the mutual distance of two level surfaces is of uniform value. What is really constant is the kinetic energy acquired by a body in falling from the upper level to the lower by any path.

So, in extension of the third order, a contour may be the intersection of a surface by a surface over which some quantity is constant.

3. This idea may be extended to objects of other dimensions than the second, and extension of other order than the third. We may state generally that the *contour of an object of* n *dimensions, existing*

*in extension of the* (n + 1)th *order, is the intersection of it with an object of* n *dimensions at every point of which some quantity is constant.* It is, therefore, of $(n - 1)$ dimensions. But, since in extension of the $(n + 1)^{\text{th}}$ order, we have objects of all positive dimensions up the $(n + 1)^{\text{th}}$ inclusive; in such extension, we may have contours of all positive dimensions up to the $n^{\text{th}}$ inclusive. The contours of a curve are points; of a surface, curves; of a solid, surfaces; of a four-dimensional object, solids; and so on. Hence, in ordinary extension we may have point, curve, and surface, contours.

4. Since space is of three dimensions, a point in space may have three degrees of translational freedom,—it may be free to move in any three mutually perpendicular directions successively or simultaneously. We are said to introduce a degree of constraint when we take away one of these degrees of freedom. If we prevent the exercise of one kind of freedom, unless a second is simultaneously exercised, we introduce a degree of constraint in the most general way possible. The moving point is then forced to remain on some definite surface, and the nature of the constraint is best shown by writing down the equation to the surface, the most general being of the form $f(x, y, z) = 0$. If another such constraint be imposed, the moving point must remain in positions common to both the corresponding surfaces. Hence, it must move in a definite curve. Three such equations determine the position of a point. They may degenerate into $f_1(x) = 0$, $f_1(y) = 0$, $f_1(z) = 0$, where $f_1(x)$ is some function of $x$ of the first degree, &c.; that is to say, the position of the point is determined by the intersection of three planes parallel to the co-ordinate planes.

5. A *curve* being of one dimension, its contours are *points*,—the points of intersection of it with curves along each of which some quantity is constant, although it differs in magnitude from one curve to another. These points may be projected upon any line in the plane $(x, y)$.

We shall consider first a plane curve, and let its plane be taken as that of $(x, y)$. We have the equations $f_n(x, y) = 0$ and $f_1(z) = 0$, where the suffix denotes the degree of the equation. Instead of the second equation, we may put $z = 0$, as this simply makes the plane of the curve that of $(x, y)$. The equations to the curves, along which some quantity, say $c$, is constant, may be written in the general form $\phi_n(x, y, c) = 0$. In the different curves of the system, $c$ has different values. As a particular example, the curves might be circles of different radii. Again, the equation might be $\phi_1(y, c) = 0$. This represents a series of lines parallel to the axis of $x$. It furnishes the simplest case for consideration, and, at the same time, the most useful.

From the reference already made (§ 2) to contour maps, it is evident that the mutual variation of three quantities can be represented upon a plane surface by means of lines. So, in the case considered, the mutual variation of two quantities can be represented upon a right line. The curve in fig. 20 is intersected by lines parallel to the axis of $x$, and the points of intersection are projected upon the axis, and designated by numbers giving the various values of $y$ corresponding to the values of $x$. If the curve be continuous a *maximum* or *minimum* value of $y$ exists between two equal values. It is a maximum if, as $x$ increases from its least value corresponding to the given value of $y$, $y$ first increases and then diminishes. It is a minimum if $y$ first diminishes and then increases. The *steepness of slope* is shown by the closeness of the contours for equal increments of $y$, and its *direction* is shown by the order in which the values of $y$ occur as regards numerical magnitude when $x$ increases. Similarly, the existence of a *maximum-minimum* value is indicated, and so on.

The contours of a closed oval and of a figure-of-eight curve, are shown in figs. 21 and 23.

6. Hitherto we have considered only plane curves. In the case of tortuous curves, it will be most convenient to obtain the contours by cutting the curve by surfaces over which some quantity is constant. As a particular case, these surfaces may be planes perpendicular to the $z$-axis, the equations to which may be written in the general form $f_1(z, c) = 0$. Obviously, the position of a moving point in space is representable by a tortuous curve. Its position at any time can be got from the curve, if the value of the time in terms any one of the co-ordinates, say $z$ is known, for we should then only have to cut the curve by the plane ${}_1f_n(z, t) = 0$, where $t$ represents the time, and ${}_1f_n$ is a functional symbol showing that the equation is of the first degree in $z$ but may be of any degree in $t$. This condition is rendered necessary by the fact that the point must be in one definite position at a given time, but may occupy the same position at different times. If we consider a number of such curves traced out in space by moving material points, we can, by cutting the curves by planes corresponding to a definite time and projecting the points so got upon any co-ordinate plane, obtain the *diagram of configuration*

of the material system at that time. It is evident that, in general, the plane corresponding to a definite time will be different for each curve. The disadvantage so entailed may be got rid of by using trilinear co-ordinates to indicate the position of the point when the time is given. If the curve be cut by any plane, the distances of the point of intersection from three intersecting lines in that plane give the $x$, $y$, and $z$ co-ordinates at an instant. This instant may be determined by the distance of the plane from a given fixed plane parallel to it. The curves, which are now taken to represent the positions of the points, are not the actual curves traced out by the points in their motion through space. But the diagram of configuration obtained from them has the advantage of showing at once the relative values of all the co-ordinates of all the points, whereas, in the Cartesian system this could not be done without projecting on all three co-ordinate planes. The triangles of reference will be similar whatever be the plane by which the curves are cut, but they will not necessarily be of the same magnitude. The similarity is necessary in order to give the proper *ratios* of the co-ordinates, the different magnitude is necessary in order to give the proper *values* of the co-ordinates. Fig. 22 shows the contours of curves $a$, $b$, by planes corresponding to times $t_1$, $t_2$.

By the aid of this diagram, the *diagram of total displacements* in a given time may be constructed. And by taking the displacements in $\frac{1}{n}$*th* of the unit of time and magnifying them $n$ times, the *diagram of velocities* can be got. Similarly, the diagrams of *acceleration*, *forces*, and so on, may be represented as the contours of curves. The curves, from which the diagrams of velocities, &c., are obtained, are, of course, different from the original curves representing the position of the moving points. In the case of velocities, for example, they are the hodographs of the original curves on this trilinear system of reference.

*Third Meeting, January 8th,* 1886.

Dr Ferguson, F.R.S.E., President, in the Chair.

## Two Mechanical Integrators or Planimeters.

By A. Y. Fraser, M.A., F.R.S.E.

### The Measurement of Areas.

For the measurement of a plane area, bounded by an irregular curve, various methods are adopted. Besides the well-known methods of approximation in use among land measurers, the following may be mentioned—

The area is divided by parallel ordinates. These are measured, and the results are then treated in several ways to produce more or less accurate results. See Williamson's *Integral Calculus*, third edition, p. 211.

The area may be drawn on paper ruled in small squares ("plotting paper"), or a sheet of glass ruled in squares is placed over the area. The complete squares included are counted, and the incomplete squares estimated.

The area is drawn on cardboard or sheet metal, cut out, and then weighed. The weight of a known area is then ascertained, and the required result got. This method gives very good results. A variation of the same method is to cover the area with small shot, which is weighed or counted.

A planimeter or mechanical integrator may be used. Here a pointer has merely to be taken round the boundary, and the instrument records the area traced out. Many planimeters have been invented; but for all practical purposes, the beautiful little instrument invented by Prof. Amsler thirty years ago, holds its own against all comers.

### Planimeters.

The problem, in general, attempted by planimeters, will be understood by a reference to figure 24. While the tracing point of the instrument moves over any arc $aa'$, an index records the area $aa'\nu'\nu$. Suppose, now, the pointer starts at A, the extreme right of the figure, and passes round ABC to C; the area recorded will be ABCEF. Suppose, again, that in bringing the pointer round CDA it is arranged that the reading given by the index is subtracted from, and not added to, the former reading (and the instrument can easily be made to effect this), then the area recorded, when the pointer has returned to A, will be ABCEF – CDAFE, that is the area ABCD.

This in fact is the mechanical integration of $\int ydx$ between the limits $x = \text{OF}$ and $x = \text{OE}$.

It is a matter of indifference where the commencement with the tracing point is made, so long as the point is stopped at the same place.

For a full discussion of the problem to be solved by the inventor of a planimeter, the reader may be referred to the papers of John Sang* and Clerk Maxwell,* and, especially to the full account of the whole matter by Professor Shaw.† Clerk Maxwell points out the division of planimeters into two kinds—(1) Those that involve slipping in some of their parts; (2) Those that involve rolling only. He enforces the objection to the former class (errors on account of friction), and gives a design for an instrument of the second kind—a design which, so far as I know, never came to anything practically.

Of the two instruments to be described here, the first may be taken to belong to the first class, that is, it involves rolling and slipping in its movements, while the second belongs to the second or purely rolling class. While it is not for a moment supposed that these can compete as practical instruments with Amsler's, they are brought forward because they are of an entirely new design, and both are of considerable mathematical interest.

## THE SINGLE ROLLER PLANIMETER.

The essential portion of this instrument is a tapering roller or spindle (fig. 25) mounted to turn on an axis, and made to turn by placing the paper, on which the area is traced, on the surface of the roller, pressing the paper against the surface with a pencil or other pointed object, and pulling or pushing the paper in a direction at right angles to the axis of the roller, and tangentially to the surface.

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* Transactions of the Royal Scottish Society of Arts, Vol. IV., p. 119; p. 420.

† Journal of the Institute of Civil Engineers, Vol. lxxxii., 1884-85, Part IV., pp. 75-164; on "Mechanical Integrators."

The more easily to understand this motion, the reader may place on a table in front of him a tumbler on its side with its bottom towards him, lay a sheet of paper over the tumbler and press it gently down on the tumbler with, say, a finger nail of the right hand. Then on pulling the paper away to the left with the left hand, the tumbler will roll along the table. If the tumbler could be mounted to rotate in the same way, about an axis passing longitudinally through it, the resemblance would be complete.

The dimensions of the spindle, of which a section is shown in fig. 25, as actually constructed are: Length GH, measured along the surface, 6 inches; circumference at H, 10 inches; at G, 2½ inches. The circumference at any point K is got thus: Let O be a point 2 inches from H; measure the distance along the surface from K to O, $n$ inches say, that is, $n - 2$ inches from K to H, then the circumference at K is $20/n$ inches.

The method of mounting the roller will be seen from figures 26, 27. LT is a box of ⅝-in. mahogany. Fig. 26 shows the box as looked at when about to be used; fig. 27, the back of the box through which the wide end of the roller, which is graduated, is seen. The top of the box is covered with two slips of tin, R, S, which slide in from each end, and do not meet in the middle by about half an inch. The tins are bent to the shape of the roller, and through the middle opening of half an inch, the roller projects so as to show a strip very slightly above the level of the lids. A guiding strip of wood is fastened from V to W, so that a pencil pressed vertically down on the roller and against this strip of wood, will always be vertically above the axis of the roller. The dimensions of the box are LM 16 ins., LN 5 ins., QP 3½ ins., NP 7 ins.

To find the area of any surface, we proceed as follows:—The roller is first turned round so that the zero point of the scale (fig. 27) may be under the pointer. A small mirror laid over against the box may be used to keep the scale always in view. Suppose the area in fig. 24 is to be integrated. The slip of paper on which the area is traced is passed along the top RS of the box, under the guide VW. The paper is placed so that A can be brought under the point of a pencil resting against the guide VW. The pencil is then put down so as to press the paper gently against the surface of the roller. The paper is pushed away to the right, the projecting ledge QY of the box, fig. 26, guiding the paper, the pencil all the time being slid up and down against the guide VW so as to follow the curve ABC. When the pencil has reached C, the area ABCEF has been recorded on the scale. The paper is now pulled to the left, and the pencil is made to follow CDA, the roller meanwhile rotating backwards, and the indicator recording the area CDAFE in the negative sense. The indicator has now recorded the area ABCFE – CDAFE, that is the area ABCD; the result being given in square inches, and whatever fraction of an inch the scale may be made to show. We shall now prove that the result is as stated.

If, while the paper lies on the top of the box, a line (OF, fig. 24) be drawn on the paper parallel to $\lambda\mu$ (fig. 26), and two inches beyond it, that is, through the point O of figure 25, we can prove that the instrument, in passing over the arc $aa'$, records the area $aa'\nu'\nu$. Suppose the length of the arc $aa'$ is $s$, and that it is $n$ inches from OF; the area of $aa'\nu'\nu$, if $s$ be taken small, will be $sn$ inches. To prove that the reading on the scale is $sn$.

The roller immediately below $aa'$ (at K, say, fig. 25) is $20/n$ inches in circumference, and $s$ inches of this circumference passes round in tracing the arc $aa'$. Now $s$ inches is $s/(20/n)$ of the whole circumference at that place, that is $sn/20$ths, and in making this movement there will be $sn/20$ths of a complete revolution of the roller. But in a whole revolution the index records 20 inches (fig. 27), therefore in making $sn/20$ths of a revolution the record will be $(sn/20) \times 20$ inches, that is $sn$ inches. And $aa'$ is any portion of the boundary, so that the proof is complete.*

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* Strictly speaking, perhaps, the arc $aa'$ should be considered as the hypotenuse of a right-angled triangle, having one side $s$ parallel, the other $t$ perpendicular to OF. The pointer would then be supposed to be taken over $s$ during which the roller would rotate (with the result as above), then over $t$ during which there would be no rotation. Considered in this way the total area accounted for is less than the area ABCD by the sum of the elementary triangles round the curve (if we suppose the triangles always made to the inside of the figure); and the length of the path traversed by the pointer is greater than the curve ABCD by a corresponding sum of small lengths. Finally, if the triangles be taken small, the limit to the areas thus considered is the area ABCD, and at the same time the limit to the curve as thus traced is the curve ABCD; therefore in passing round ABCD the area integrated is the area ABCD.

Of course instead of 20 as the fundamental constant for the roller, any other quantity might have been taken. The part of the curve also which is taken for the spindle may be varied, only at the low end it must be steep enough to show a change in the readings, and at the other end it must not be inconveniently steep.

The general equation in rectangular co-ordinates to the curve GKH (fig. 25), can be determined from the defining property, which may be put as follows:—Let the arc $\text{OK} = s$ then the circumference at K is $20/s$, say $c/s$. If $y$ be the ordinate at K we have $y = c/2\pi s$, from which the equation to GKH can be found in terms of $x$ and $y$.

The general result, which I get in what for the part of the curve to be used is a convergent series of powers of $y$, is not of any interest. The particular equation, terms beyond a certain distance being omitted, from which the actual curve under discussion was traced is

$$x = \frac{c}{y} + \frac{1}{6}\frac{y^3}{c} + \frac{1}{56}\frac{y^7}{c^3} - 2{\cdot}2258$$

where $c = 10/\pi = 3{\cdot}183$.
The curve reduces to an equilateral hyperbola whenever the terms after the first can be neglected.

Though there is slipping in this instrument in the working parts, that is, between the pencil and the paper, there is none where it would be of any consequence, namely, between the paper and the roller.

The difficulty with this planimeter, beyond the difficulty of constructing the roller, which can be got over well enough, is in co-ordinating the movements of the right and left hands so that the curve is accurately followed. This difficulty of avoiding small departures from the curve, led to the attempt to devise a modification of the instrument, where the paper should rest on a table and a pointer could be guided over the curve with a single motion. This attempt resulted in the construction of a much more interesting planimeter, of which a brief account will now be given.

## THE DOUBLE ROLLER PLANIMETER.

The arrangement of this instrument is shown in figure 28, which is an outline sketch of the principal parts. The motion is imparted to the lower spindle by the upper spiral roller, which is pivoted about H and G. The axis HG is kept always in the same horizontal plane by means of the two supports GC, HC′, running on castors, and the pointer P, which is rigidly attached to the frame FFF, carrying H, G; and HG is kept always parallel to itself and at right angles to the axis of the lower spindle by means of a linkwork contrivance (a pair of parallelograms) not shown in the figure. While the pointer P traces out the area, the spiral roller moves about, subject to the restrictions just mentioned, and acts as a driver of the spindle; the connection between the two being always one of *pure rolling contact.*

Inasmuch as the area to be integrated has not now to be bent along the surface of the roller, but lies in a horizontal plane, the different circumferences of the under roller are determined not now by distances measured along its surface, but by distances measured along its axis. This change being made, the numbers already given for the last spindle will apply here. The most interesting change is in the equation to the curve. For instead of $y = c/2\pi s$ we have $y = c/2\pi x$ that is

$$xy = \text{ a constant,}$$

and therefore the curve is an equilateral hyperbola.

The proof that this instrument does integrate the area is so very little different from that already given that it may be dispensed with here.

A little consideration will show that it is theoretically indifferent whether the spiral roller move with the pointer, the other roller being fixed, or the under roller move with the pointer while the spiral is fixed.

The spiral roller has one peculiarity not yet touched on. In order to secure that when the pointer is moved back an inch say, the point of contact of the two rollers should also move back the same horizontal distance, it is arranged that the point of contact is in all positions of the rollers vertically below the axis HG. This property makes the finding of the equation to the spiral a very interesting problem. We may conclude with the discussion of this problem.

## To find the Equation to the Spiral.

As S (fig. 29) moves along the line $y = b$, the spiral, whose centre is S, rolls along the hyperbola $xy = c^2$ so as to have the point of contact Q always vertically below S.

Let $x$, $y$ be the rectangular co-ordinates of Q considered as a point on the hyperbola, and $r$, $\theta$ the polar co-ordinates of the same point considered as belonging to the spiral.

Then $QN = rd\theta = dx$;

$\mathrm{PN} = dr = -dy$ (got either from the figure directly or from $y = b - r$),

therefore $$\frac{rd\theta}{dr} = -\frac{dx}{dy} \qquad (1).$$

But $$xy = c^2$$

therefore $$\frac{dx}{dy} = -\frac{c^2}{y^2} = -\frac{c^2}{(b-r)^2}$$

whence substituting in (1) we get

$$\frac{rd\theta}{dr} = \frac{c^2}{(b-r)^2} \qquad (2);$$

therefore $$\int d\theta = c^2 \int \frac{dr}{r(b-r)^2}$$

$$= c^2 \int \left\{ \frac{1}{b^2 r} + \frac{1}{b^2(b-r)} + \frac{1}{b(b-r)^2} \right\} dr;$$

therefore $$\theta = c^2 \left\{ \frac{1}{b^2}\log r - \frac{1}{b^2}\log(b-r) + \frac{1}{b(b-r)} \right\} + k$$

$$= \frac{c^2}{b^2}\log\frac{r}{b-r} + \frac{c^2}{b(b-r)} + k \qquad (3)$$

$$= \frac{c^2}{b^2}\log_e 10 \log_{10}\frac{r}{b-r} + \frac{c^2}{b(b-r)} + k$$

$$= \frac{c^2}{b^2}\left\{ 2{\cdot}3049\log_{10}\frac{a}{1-a} + \frac{1}{1-a} + \lambda \right\} \qquad (4);$$

where $2{\cdot}3049 = \log_e 10$

$$a = \frac{r}{b}$$

$$\lambda = k \div \frac{c^2}{b^2}.$$

Equation (3) is the general equation to the spiral; (4) is the more convenient form from which to trace the curve.

The spiral is seen to have an infinite number of windings about the point S, and to be asymptotic to a circle of radius $b$.

In the spiral as actually constructed the constants had to be chosen so as to be practically convenient, and after a number of trials, TA (fig. 29), that is the shortest radius of the part of the spiral to be used, was taken equal to $\frac{1}{5}b$, and $b$ is therefore $\frac{5}{4}$ of AB, which again is the radius of a circle 10 inches in circumference. The equation actually used for tracing the spiral is

$$\theta = 1{\cdot}8533\log\frac{a}{1-a} + {\cdot}8041\frac{1}{1-a} + {\cdot}1107 \qquad (5)$$

Going from $r=8b/40$ by fortieths of $b$ up to $r=32b/40$, using equation (5), and changing from circular measure to degrees, we get the following values:—

| $r$ (inches) | $\theta$ (degrees) | $\Delta\theta$ | $r$ (inches) | $\theta$ (degrees) | $\Delta\theta$ |
|---|---|---|---|---|---|
| ·398 | 0·00 | | | | |
| ·448 | 8·74 | 8·74 | 1·044 | 107·95 | 9·46 |
| ·497 | 17·11 | 8·37 | 1·094 | 117·98 | 10·03 |
| ·547 | 25·19 | 8·08 | 1·144 | 128·74 | 10·76 |
| ·597 | 33·09 | 7·90 | 1·194 | 140·21 | 11·47 |
| ·647 | 40·89 | 7·80 | 1·243 | 152·76 | 12·55 |
| ·696 | 48·68 | 7·79 | 1·293 | 166·52 | 13·76 |
| ·746 | 56·50 | 7·82 | 1·343 | 181·81 | 15·29 |
| ·796 | 64·43 | 7·93 | 1·393 | 198·99 | 17·18 |
| ·846 | 72·52 | 8·09 | 1·442 | 218·58 | 19·59 |
| ·895 | 80·86 | 8·34 | 1·492 | 241·28 | 22·70 |
| ·945 | 89·48 | 8·62 | 1·542 | 268·15 | 26·87 |
| ·995 | 98·49 | 9·01 | 1·592 | 300·64 | 32·49 |

Only one remark will be made on the above table. As $r$ increases uniformly the column for $\Delta\theta$ shows that $\theta$ is not increasing uniformly, and that in fact there is a minimum value for $d\theta/dr$ between $r=13b/40$ and $r=15b/40$. It may be interesting to show that this can be deduced from one of our equations.

From (2) we have

$$\frac{d\theta}{dr}=\frac{c^2}{r(b-r)^2}=\frac{c^2}{b^2r-2br^2+r^3};$$

therefore

$$\frac{d^2\theta}{dr^2}=-c^2\frac{b^2-4br+3r^2}{r^2(b-r)^4}=-c^2\frac{(b-3r)(b-r)}{r(b-r)^4}$$

$$=-c^2\frac{b-3r}{r^2(b-r)^3} \qquad (6)$$

Equating (6) to zero to find maximum or minimum values for $d\theta/dr$ we have $b-3r=0$, that is $r=\frac{1}{3}b=13\frac{1}{3}b/40$ which accords with the result in the table.

# Solution of a Problem proposed by Dr Muir.

## By R. E. ALLARDICE, M.A.

The problem is to show that the expression

$$\{(a^{\frac14}+b^{\frac14})^4-c\}\{(a^{\frac14}+\omega b^{\frac14})^4-c\}\{(a^{\frac14}+\omega^2 b^{\frac14})^4-c\}\{(a^{\frac14}+\omega^3 b^{\frac14})^4-c\},$$

where $\omega$ is an imaginary fourth root of unity, is symmetrical with respect to $a$, $b$, and $c$.

*First Method.*—Denote the above expression by P. Then, by means of the identity $x^4-y^4=(x+y)(x+\omega y)(x+\omega^2 y)(x+\omega^3 y)$, P may be expressed as the product of sixteen factors, namely, the sixteen values of $a^{\frac14}+\omega^r b^{\frac14}+\omega^s c^{\frac14}$, where $r$ and $s$ have successively the values 1, 2, 3, 4. This product is obviously symmetrical, and the symmetry is not destroyed to the eye by a re-combination of the factors in sets of four, the form thus obtained being

$$(a+b+c-2a^{\frac12}b^{\frac12}-2b^{\frac12}c^{\frac12}-2c^{\frac12}a^{\frac12})(a+b+c-2a^{\frac12}b^{\frac12}+2b^{\frac12}c^{\frac12}+2c^{\frac12}a^{\frac12})$$
$$\times(a+b+c+2a^{\frac12}b^{\frac12}-2b^{\frac12}c^{\frac12}+2c^{\frac12}a^{\frac12})(a+b+c+2a^{\frac12}b^{\frac12}+2b^{\frac12}c^{\frac12}-2c^{\frac12}a^{\frac12}).$$

*Second Method.*—Since

$$(a^{\frac14}+\omega b^{\frac14})^4-c=\omega^{12}(a^{\frac14}+\omega b^{\frac14})^4-c=(\omega^3 a^{\frac14}+b^{\frac14})^4-c,$$

P is evidently symmetrical with respect to $a$ and $b$. Further, since the change of $b^{\frac14}$ into $\omega b^{\frac14}$ does not alter P, P is rational in $b$ and therefore also in $a$. When $b=0$, P becomes $(a-c)^4$ which is symmetrical in $a$ and $c$. Hence P has been proved symmetrical in $a$ and $b$, and symmetrical in $a$ and $c$ except in those terms which contain all the three letters. Thus

$$P=\Sigma a^4+A\Sigma a^3b+B\Sigma a^2b^2+C(a^2bc+ab^2c)+Dabc^2.$$

It remains to prove $C=D$.

The co-efficient of $c^2$ in P is $\Sigma(a^{\frac14}+b^{\frac14})^4(a+\omega b^{\frac14})^4$ and D is equal to the value of this when $a=b=1$, less twice the value when $a=1$, $b=0$, that is $-112-2\times6=-124$.

The co-efficient of $c$ in P is $-\Sigma(a^{\frac14}+b^{\frac14})^4(a^{\frac14}+\omega b^{\frac14})^4(a^{\frac14}+\omega^2 b^{\frac14})^4$ and 2C is equal to the value of this when $a=b=1$, less twice the value when $a=1$, $b=0$, that is $-(256-2\times4)=-248=2D$.

Therefore $C=D$.

*Fourth Meeting, February 12th,* 1886.

Dr FERGUSON, F.R.S.E., President, in the Chair.

## Kinematical Theorems.

### By WILLIAM HARVEY, B.A.

### I.—Motion of a plane figure in its plane.

§ 1. Figure 30. Let AB be any straight line in the plane figure, A′B the position of the same line after a displacement, the point A moving to A′ and B to B′. Since the position of a plane figure in its plane is determined, when the position of a straight line rigidly attached to it is determined, the motion of the plane figure is determined if we determine the motion of the straight line AB.

Now, it is known that the straight line AB may be brought into the position A′B′ by a rotation about a fixed point O, in the general case at a finite distance. It is proved that the point O is the intersection of two straight lines, one drawn through the middle point of AA′ perpendicular to AA′, and the other drawn through the middle point of BB perpendicular to BB′. This follows, very simply, from the equality of the triangles OAB and OA′B′. The point O so determined is unique.

Further, the rotation about the point O is measured by the angle AOA′ or BOB′, each of these angles being equal to the angle between the lines AB and A′B′.

If we define the initial and final positions of any point of the figure as corresponding points, so that A, A′; B, B′ are corresponding points, and if we take any point P on AB and the corresponding point P′ on A′B′, then the rotation about O which brings AB into the position A′B′, will clearly bring the line AP into the position A′P′.

§ 2. Hence, by the theorem referred to, the straight line through Q, the middle point of PP′, perpendicular to PP′, passes through O, and PP′ subtends an angle POP′ at O equal to the angle of rotation of the plane figure about O. And the triangles AOP and A′OP′ are equal in every respect. These propositions hold for any pair of corresponding points.

§ 3. We shall denote the intersection of the lines AB, A′B′ by two letters D or C′, for a reason which will be obvious immediately. On the straight line A′B′, take A′D′ equal to AD; then D and D′ are corresponding points. Again, on AB take AC equal to A′C′; then C and C′ are corresponding points. And it is obvious that CC′ is equal to DD′. It follows that the straight line drawn through E, E′, the middle points of CC′ and DD′ perpendicular to these lines respectively, pass through O. It is further evident that the angle EOE′ is equal to the angle of rotation, that E and E′ are corresponding points, and that OD bisects the exterior angle between the lines AB, A′B′.

§ 4. Again, since the angle OPD is equal to the angle OP′D, a circle can be described to pass through the four points O, D, P, P′. But if, from a point O in the circumference of a circle, perpendiculars be let fall on the sides of a triangle inscribed in the circle, the feet of these perpendiculars lie in a straight line, which is equally distant from O and from the orthocentre of the inscribed triangle. This line is called the pedal line of the triangle. Hence, the points E, E′ and Q lie in one straight line, the pedal line of the triangle PDP′.

Further, if we take any other pair of corresponding points on AB and A′B′, R and R′, then the middle point of RR′ lies on the same line. Hence the locus of such points is a straight line.

§ 5. If we regard O as the focus, and the line EE′Q as the tangent at the vertex of a parabola, then the line OQ is drawn from the focus, and meets the tangent at the vertex in Q, and PP′ is drawn at right angles to OQ, and, therefore, PP′ touches this parabola. Further, if any other pair of corresponding points R, R′ be taken on AB, A′B′ respectively, then RR′ will touch the same parabola. Hence, if corresponding points be taken on AB, A′B′, the lines joining them envelope a parabola. The directrix of this parabola is the locus of the orthocentres of triangles, formed by the lines joining corresponding points and the lines AB, A′B′.

When the motion is infinitesimal, this theorem becomes: If any plane figure is in motion in its plane, the trajectories of points situated, at any instant in a straight line, envelope a parabola.

## II.—Motion of a solid body in space.

I intend, in this division, to give two theorems for the representation of the finite displacement of a solid body in space, and to illustrate the geometry of displacement by a few deductions from these theorems, analogous to those I have given in two dimensions.

The first of these theorems is given by Chasles in the *Comptes Rendus de l' Académie des Sciences* for 1843 for the case of infinitesimal motion, and the proof he gives is indirect:—

§ 1. Figure 31. The theorem is as follows:—Any finite displacement of a solid body in space may be represented, in an infinite number of ways, by two successive finite rotations about two axes, at right angles to each other, and, in general, not intersecting.

Since the position of a solid body is determined when the position of a plane rigidly attached to it is determined, the displacement of such a body will be determined, when that of the plane is determined.

Let CAB be any such plane, C′A′B′ the position of the same plane after the displacement. Let A′B′ be the intersection of the two positions of the plane, and let AB be the line in the plane CAB corresponding to A′B′ in C′A′B′, so that A, A′; B, B′ are corresponding points. It is clear, if we except the case of motion in a plane, that the only points which remain in the plane after the displacement, and whose displacements are entirely in the plane, are the points in the line A′B′. Now AB is the initial position of A′B′, and, from what has been proved of motion in a plane, it is clear that AB may be brought into the position A′B′ by a finite rotation about an axis perpendicular to the plane CAB, and, therefore, to the line A′B′. By this rotation, a line AB of the solid body in its initial position, is brought into co-incidence with a line A′B′ in its final position, and it is evident that the displacement may be completed by a simple rotation about A′B′ regarded as a fixed line. Hence any solid body may be brought from one position in space to another by two rotations, about two axes at right angles to one another. Such rotations are called conjugate rotations, and the axes conjugate rectangular axes. The order of the rotations is evidently important, and there will be a pair of such axes corresponding to every plane in the body.

§ 2. Figure 32. When the motion is represented by two rotations as above, there is a point on each axis whose motion is entirely along that axis.

Let A′B′, C′D′ be the axes, A′X the shortest distance between them, and let C′D′ be the axis perpendicular to the base plane, and A′B′ the axis in it.

At the point A′ in the straight line A′X, make the angles XA′S, XA′S′ on opposite sides of A′X, in the plane A′XD′, each equal to the semi-angle of rotation about A′B′, and let S, S′ lie on C′D′. By the rotation about C′D′, the position of S is unaltered, and by the subsequent rotation about A′B′, S is clearly brought to S′. Hence, the axis C′D′ joins the corresponding points S, S′. Again, if we take T, T′ on A′B′ obtained in a similar way, then by the rotation about C′D′, T is brought to T′, and the subsequent rotation about A′B′ will not effect its position. Hence, the axis A′B′ joins the corresponding points T, T′.

From this, it follows that every line joining two corresponding points is one of a pair of conjugate rectangular axes.

§ 3. Figure 32. The line joining any two corresponding points, is also the intersection of two corresponding planes.

Let C′D′ be a line joining two corresponding points S and S′, and let CD be the initial position of C′D′. Then CD evidently cuts C′D′ in the point S, and, therefore, CD and C′D′ are in the same plane, and therefore, as above, CD may be brought to the position C′D′ by a finite rotation about an axis perpendicular to this plane. (It is to be observed that this axis is not A′B′). Hence the whole displacement of the body may, as before, be accomplished by a rotation about the axis perpendicular to the plane containing CD and C′D′, followed by a rotation about C′D′ regarded as a fixed line. Hence C′D′ is the intersection of two corresponding planes. And we have seen that A′B′ is also the intersection of two corresponding planes, and, hence the proposition follows whether the axis considered is that round which the first or second rotation takes place. That is to say, the lines joining S, S′ and T, T′ are the intersections of corresponding planes, and the same is true for every line joining corresponding points.

§ 4. If we consider any two positions of a solid body and join the corresponding points, the lines which lie in one and the same plane envelope a parabola.

As usual, let us represent the motion by a rotation about an axis perpendicular to the plane, followed by a rotation about an axis in the plane.

Let CD and A′B′ be these axes, and let AB be the initial position of A′B′, then we have shown that the only points whose displacements are entirely in the plane, are the points lying in the line AB. And hence from the last theorem on motion in two dimensions we see that the lines joining the different points of AB to the corresponding points of A′B′ envelope a parabola.

This theorem may also be stated as follows:—The lines of intersection of corresponding planes which lie in one and the same plane, envelope a parabola.

§ 5. Figures 33, 34. If we consider any two positions of a solid body and join the corresponding points, the lines which pass through one and the same point form a cone of the second order, and the points themselves form, on this cone, a curve of the third order.

This theorem is given by Chasles for the case of finite as well as infinitesimal displacements in the *Bulletin Universel des Sciences* for 1830. I have not seen his proof. It may, however, be proved in the same way as the preceding theorems.

Let O be the given point, and let PP′, QQ′ pass through O, where P, P′ and Q, Q′ are corresponding points.

As before, the displacement may be represented by a rotation about a line perpendicular to the plane OPQ, by which PQ is brought into the position P′Q′, followed by a rotation about P′Q′ regarded as a fixed line, and the points on PQ are the only points whose displacement lies in the plane OPQ. Hence, if R be any point in the plane OPQ not lying in the line PQ and R′ the corresponding point, RR′ meets the plane in R, and therefore cannot pass though O. If R lies in PQ, then the corresponding point R′ lies in P′Q′ and PR = P′R′ as also PQ = P′Q′. Hence RR′ cannot pass through O unless P′Q′ is parallel to PQ, in which case the point O would not be at a finite distance. Hence if any plane passes through O, it contains two generators PP′, QQ′ of the locus only, and therefore the locus is a cone of the second order. If the point O is at infinity, the locus reduces to a plane.

The result may also be stated as follows:—The intersections of corresponding planes which pass through one and the same point form a cone of the second order.

To prove the second part of the theorem, let OP, OQ be two generators of the cone, and let O′ be the final position of the point O, then by the hypothesis OO′ lies on the cone. (Fig. 34).

Again, if we join O′ to the points P′, Q′, &c., the final positions of P, Q, &c., we clearly get a cone equal in every respect to the cone OPQ. In fact, the cone O′P′Q′ is simply the cone OPQ moved into a new position by the displacement.

Further, if O″ be the initial position of a point whose final position is O, then OO″ lies on the cone OPQ, and after the displacement OO″ takes up the position O′O. Hence, O′O is the line joining O′ to the final position of O″, and, therefore, by the hypothesis lies on the cone O′P′Q′. Hence the generator OO′ is common to the two cones. Therefore, the curve P′Q′, &c., is the intersection of two cones of the second order, which have a generating line in common, and therefore is of the third order, since the complete intersection of the two cones is of the fourth. The curves PQ &c., and P′Q′ &c., are evidently the same, whence the second part of the theorem follows.

§ 6. The following theorem connects the two preceding. If any number of axes pass through a point, their respective conjugates lie in a plane. Let O be the given point, and let O′ as before be the final position of O. A rotation about any axis through O will not affect the position of O, hence the displacement OO′ is entirely due when rotation about the conjugate of the particular axis chosen. But O can be brought to O′ only by a rotation about an axis lying in a plane bisecting OO′ at right angles, because O and O′ must be equi-distant from every point of the axis. Hence whatever axis through O is chosen, its conjugate must lie in this plane, which is therefore the locus of the conjugates.

§ 7. We may state the three preceding results in one proposition as follows:—If the displacement of a body be represented by two conjugate rotations about axes at right angles to each other, then if any number of axes pass through a point they form a cone of the second order, while their respective conjugates lie in a plane and envelope a parabola.

§ 8. Figures 35, 36. We come now to the second theorem, which is a more general case of the first. In the first case we may choose any line joining two corresponding points as an axis, its conjugate then lies in the plane bisecting the line joining the corresponding points at right angles, and is the intersection of this plane with its final position after the displacement. The axis in the plane is called by Chasles the characteristic of the plane. We see then, that in the first case the choice of the axes is not altogether arbitrary.

The more general theorem is:—Any given finite displacement of a solid body in space may be represented by two finite rotations, one about any given straight line and the other about another straight line which does not generally intersect the first.

A demonstration of this theorem is given by Rodrigues in Liouville's *Journal de Mathématiques* for 1840, but it depends on the laws of composition of finite rotations. Mr Routh, who also gives the theorem, follows Rodrigues. The following is a direct demonstration:—

Let AB be the initial, and A'B' the final position of any straight line in the body. We shall show that the straight line may be brought from the position AB to the position A'B' by a rotation about a certain axis.

Let A, A', B, B' be corresponding points. Through the middle point of AA' draw a plane perpendicular to AA', and through the middle point of BB' draw a plane perpendicular to BB'. These planes intersect in the required axis. Through AA' draw the plane AA'O perpendicular to the axis, and through BB' draw BB'O' perpendicular to the axis, and let these planes meet the axis in O, O' respectively. Then since O lies in the plane bisecting AA' at right angles, $OA = OA'$; and since O lies in a plane bisecting BB' at right angles $OB = OB'$ similarly $O'A = O'A'$ and $O'B = O'B'$, also $AB = A'B'$. Hence the six edges of the tetrahedron OABO' are respectively equal to the six edges of the tetrahedron OA'B'O', and therefore the tetrahedra are equal in every respect. Therefore the angles between their opposite edges are equal. Whence the angle between OA and O'B is equal to the angle between OA' and O'B' and the planes OAA', OBB' are parallel. It follows from this that the angle AOA' is equal to angle BOB', and that the rotation about OO', which brings A to A', brings B to B', and therefore the straight line AB to A'B'.

The axis OO' is perpendicular to a plane containing two straight lines parallel to AA', BB', respectively. This property is included in the following proof. (Fig. 36).

Let AB, A'B', be the two positions of the line, so that $AB = A'B'$.

Through AA' draw a plane parallel to BB' and let BP, B'P' be perpendicular to this plane. Then $BP = B'P'$.

Also $AP = \sqrt{AB^2 - PB^2} = \sqrt{A'B'^2 - P'B'^2} = A'P'$. Therefore by a rotation about an axis perpendicular to the plane APA' we may bring A to A', P to P', and, therefore, B to B'.*

If, then, we rotate the body about the axis so determined, the line AB in the initial position of the body is brought into its final position A'B', and the displacement may evidently be completed by a second rotation about A'B' regarded as a fixed line. Here we see one of the axes, A'B', is arbitrary, but the second is known as soon as the first is determined.

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* This proof was suggested to me by a friend.

§ 9. If the straight line AB is parallel to A′B′ then for one of the rotations we may substitute a translation, and the change of position is accomplished by a translation and a rotation. Again if AB and A′B′ coincide in situation, without the corresponding points in each coinciding, then the change of position is accomplished by a translation along and a rotation about the same axis. I here assume Chasles' celebrated results, published in 1830 in the *Bulletin Universel des Sciences*, viz:—If any finite displacement be given to a free solid body in space, there exist in the body a set of straight lines, which, after the displacement, are parallel to their initial positions, and in particular there is one straight line, and one only, which after the displacement remains in its original situation. All that need be pointed out at present is that the representation of the displacement of a solid body by a translation and a rotation, and the unique representation by a screw movement, are particular cases of the more general representation by two conjugate rotations.

§ 10. If we consider any two positions of a solid body in space, the displacement of the different points lying in any straight line lie on an hyperbolic paraboloid.

For let AB be the given line, CD its conjugate, then the displacements of every point in AB pass through two straight lines, viz., AB and its displaced position A′B′, and, further, they are all parallel to a plane perpendicular to CD. Hence they lie in an hyperbolic paraboloid. This proposition is given in Routh's *Rigid Dynamics* for infinitesimal motions, and the proof is exactly the same for finite motions.

## A Proof of Lagrange's Theorem.

By T. Hugh Miller, M.A.

If $y = z + xf(y)$ where $x$ and $z$ are independent, Lagrange's series gives the expansion of any function of $y$ in terms of $x$. The coefficients of the powers of $x$ may be found thus. Let $\phi(y)$ be the function to be expanded. Then by Taylor's theorem

$$\begin{aligned}\phi(y) &= \phi\{z + xf(y)\}\\ &= \phi(z) + xf(y)\phi'(z) + \frac{x^2[f(y)]^2}{\underline{|2}}\phi''(z) + \&c., \qquad (1)\end{aligned}$$

Also

$$f(y)=f(z)+xf(y)f'(z)+\frac{x^2[f(y)]^2}{\underline{|2}}f''(z)+\text{\&c.}$$

Now, by substituting the value of $f(y)$, $[f(y)]^2$ from the second series, the co-efficients of $x$, $x^2$, &c., can be found. This gives

$$\phi(y)=\phi(z)+xf(z)\phi'(z)+\frac{x^2}{\underline{|2}}[\{f(z)\}^2\phi''(z)+2f(z)f'(z)\phi'(z)]+\text{\&c.},$$

or $$\phi(y)=\phi(z)+xf(z)\phi'(z)+\frac{x^2}{\underline{|2}}\frac{d}{dz}\{[f(z)]^2\phi'(z)\}+\text{\&c.}$$

The co-efficients become more complicated, but suppose the above law of formation is true up to the $n^{\text{th}}$ term, that is,

Let $$\phi(y)=\phi(z)+xf(z)\phi'(z)+\ldots+\frac{x^{n-1}}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}[\{f(z)\}^{n-1}\phi'(z)]+\text{\&c.}$$

Then $$f(y)=f(z)+xf(z)f'(z)+\ldots+\frac{x^{n-1}}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}[\{f(z)\}^{n-1}f'(z)]+\text{\&c};$$

and $$\{f(y)\}^r=\{f(z)\}^r+xf(z)r\{f(z)\}^{r-1}f'(z)+\ldots$$
$$+\frac{x^{n-1}}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}[\{f(z)\}^{n-1}r\{f(z)\}^{r-1}f'(z)]+\text{\&c.}$$

Now substitute these values of $f(y)$, $\{f(y)\}^2$ etc. in the series (1), and find the co-efficient of $\frac{x^n}{\underline{|n}}$.

The $(n+1)^{\text{th}}$ term of $\phi(y)$ gives $\{f(z)\}^n\phi^n(z)$,
the $n^{\text{th}}$ ,, $n\phi^{n-1}(z)(n-1)\{f(z)\}^{n-1}f'(z)$
the $(n-1)^{\text{th}}$ ,, $n(n-1)\phi^{n-2}(z)\frac{1}{\underline{|2}}\frac{d}{dz}\{[f(z)]^2(n-2)[f(z)]^{n-3}f'(z)\}$
&c., ,, &c.,
the second ,, $\underline{|n}\phi'(z)\cdot\frac{1}{\underline{|n-1}}\frac{d^{n-2}}{dz^{n-2}}\{[f(z)]^{n-1}f'(z)\}$.

Therefore, the co-efficient of $\frac{x^n}{\underline{|n}}$ is

$$\{f(z)\}^n\phi^n(z)+(n-1)\frac{d}{dz}\{f(z)\}^n\phi^{n-1}(z)$$
$$+\frac{(n-1)(n-2)}{\underline{|2}}\frac{d}{dz}\left[\frac{d}{dz}\left\{f(z)\right\}^n\right]\phi^{n-2}(z)+\ldots+\frac{d^{n-2}}{dz^{n-2}}\left[\frac{d}{dz}\left\{f(z)\right\}^n\right]\phi'(z)$$

that is $\frac{d^{n-1}}{dz^{n-1}}[\{f(z)\}^n\phi'(z)]$, by Leibnitz's theorem. Therefore since the law has been proved for the first three terms, it holds universally.

## *Fifth Meeting, March 12th*, 1886.

W. J. MACDONALD, Esq., M.A., F.R.S.E., in the Chair.

## Bifilar Suspension treated by the method of Contour Lines.

### By HARRY RAINY, M.A.

The object of the following paper is to illustrate how readily some problems, which frequently occur in physical work, may be solved by the application of certain methods which are not very generally employed in mathematical investigations. The method on which the following proof is based is that of contour lines. These are a device enabling us to represent the third dimension on a plane, and are such that all the points on one contour line are at one and the same height above the level of the base plane. Probably the best known application of the principle of contours is to be found in the Survey Maps, where successive (closed) curves pass through all points at heights of 100, 200, 300, &c., feet above the sea-level; but they are of equal value in many physical diagrams, where they appear as equipotential lines, isothermals, lines of equal illumination, lines of equal force in a magnetic field, as well as in many other forms. In this paper they are employed as representing the third dimension merely. When contour lines are placed so as to indicate an equal rise in the intervals between each successive pair [or, in fact, according to any definite system], we can, with their aid, tell two things about a surface; the one—what is the steepness, or gradient, in any direction from a given point; the other—which is the direction of steepest slope at any point. The absolute steepness is measured by the amount of vertical motion per unit of horizontal motion of a point; *i.e.*, by the tangent of the angle between the horizontal plane and the line in which the point moves. This is a fact with which everyone is acquainted; every person knows what is meant by saying that the gradient along a line of railway is 1 in 50 or 1 in 500, as the case may be. To find the steepness at any point we merely need to know, in terms of the length of the line taken to represent unit steepness, the length of the line passing through the point in the given direction, and terminated by the two contours on either side of that point. The length of the line will then be *inversely* as the steepness.

The direction of steepest slope—the "Stream-line"—is that which cuts the contours at right angles. This is obvious when we consider that such a line is the shortest which can be drawn between two contours, one very close on each side of the given point and passing through the given point.

We may now proceed to apply the principles indicated above. and first we shall consider the case of the intersection of a vertical cylinder with an inclined plane (fig. 37). To save employing a rather lengthy phrase, I shall call the actual line of intersection of the cylinder with the plane, or other figure under discussion, the "path," and its representation on the contour diagram the "projected path." A point on the path and its projection are said to correspond.

In a plane the contours marking equal increments of height are equidistant, parallel, straight lines, and therefore the *direction* of maximum steepness is the same at any point in the plane, and the actual *maximum* steepness is the same for every point.

Hence, in the intersection of a plane and cylinder, *wherever* the projected path, which is obviously circular, is situated, the steepest part will always be in the same direction, and will be that which cuts the contours at right angles. It will, therefore, be situated at a point distant from that of no steepness by one quarter of the circumference of the projected path. The direction of motion at that point of the path is that of the diameter joining the highest and lowest points of the path, *i.e.*, of the shortest, and therefore the steepest line joining them.

Further, the measure of the steepness of the path at any intermediate point, in terms of the steepness at the point of maximum gradient, will be inversely as the length of the tangent to the corresponding point intercepted between two contours is to the length of the tangent at the steepest point similarly intercepted; or, if these distances be represented by $l$ and $l'$, inversely as $\frac{l}{l'}$; *i.e.*, as $\frac{l'}{l}$. But $\frac{l'}{l}$ is the sine of the angle subtended at the centre by that part which lies between the point of no steepness and the point under consideration Hence, if we call the maximum steepness unity, the steepness of the path varies from point to point, starting from that of no steepness, as the sine of the angle through which the corresponding point has turned.

The next case which may be taken as leading up to the final one is that of the intersection of a cylinder with a cone (fig. 38). In a cone the shape of the contours will, of course, be changed. They now are a series of concentric circles whose radii increase in A.P.—the common centre being the point of projection of the axis of the cone. Under these circumstances the perpendicular distance between any one pair of contours is the same as that between any other, and therefore every point has the same value for maximum gradient. Hence, when we have a cylinder intersecting a cone (both axes being supposed vertical), the projected path is steepest at its point of orthogonal intersection with the contours. The position of this point depends on the distance between the two axes, in terms of the cylinder radius. When the distance is great, the point of maximum steepness approaches that of the path traced by the intersecting plane and cylinder; when the distance is diminished, the point approaches that position of minimum (zero) steepness which is nearest the axis of the cone, and eventually, when the circumference of the cylinder cuts the axis of the cone, this point lies on the axis itself.

Lastly, if we take the intersection of a cylinder and a sphere (fig. 39), and consider it by the same method, we note that the contours of a sphere are concentric circles, having the projection of the axis (vertical) of the sphere for their common centre; but they are no longer spaced equally—they become more and more close to each other as they recede. In fact, since the steepness from point to point along the sphere varies as the tangent of the angle at the centre of the sphere between the axis and the line joining that point with the centre, the spacing of the contours varies from the centre as the cotangent of the same angle [$\theta$].

Now, it is obvious that, until the point of orthogonal intersection is reached, both the slope of the sphere and the curvature of the cylinder conspire to increase the steepness of the path, and, thereafter, the increasing slope of the sphere tends still to increase the steepness, while the motion of the point moving round its path draws it away from the line of maximum steepness,—that perpendicular to the contours,—which it had at the instant of orthogonal intersection.

But we have already seen that the increase of steepness, due to the changing slope of the sphere, varies as tan $\theta$, and also it is easy to see that the steepness of the path depends on a function of the angle through which the point turns round the cylinder. This function has a value equal to unity at the point of orthogonal intersection, and to zero at the lowest point of the path. If, then, we call this function $f(\phi)$, the steepest point on the path is that at which

$\theta \times f(\phi)$ is a maximum. This can be found when we know the relations between $\theta$ and $\phi$, which is the case when we know the radius of the sphere, the radius of the cylinder, and the distance between their axes.

If the radius of the sphere be very large compared with the radius of the cylinder, we may consider—for all practical cases—that the path is formed by the intersection of the cylinder with a plane which is the tangent plane to the sphere at the point where the axis of the cylinder cuts it.

Now, to apply these facts to bifilar suspension, we need only add to what we have already observed, the fact that the steeper a track is, the greater is the force required to drive a body of given mass up it against gravity. If we suppose a force tends to drive a body along the track of intersection of a sphere and a cylinder, or, since in all practical instruments the radius of the sphere is very large compared with that of the cylinder, the intersection of a plane and a cylinder, then by the length of the track over which the force was able to drive the body, we could measure the force. Forces of every amount from zero up to that which moves the body one quarter of the way round the track, can thus be measured by such an instrument; [all the forces less than that being, at some definite point or other, balanced by the component of gravity acting in the opposite direction *down* the path.] Such an instrument might be constructed by making a bar capable of rotating round, and sliding up and down a vertical rod; while to the further end of the bar is attached a cord whose other end is fixed to some point above the bar, and not in line of the vertical rod. In this case, evidently the lower end of the cord can move only on the surface of a sphere, having for centre the point of attachment: while, at the same time, because it is there fastened to the bar, it must move along the surface of a cylinder, and, therefore, it traces out the line common to both.

The inevitable friction involved in such a form of apparatus would, however, render it practically useless, and, therefore, the following substitute is employed. A bar has a cord attached to each end, these cords are of the same length and are fixed at equal heights above the bar, *i.e.*, in the same horizontal line. If, then, a vertical line be drawn through the centre of the bar, it will be the axis of symmetry of the system, and the bar will rotate round it and move up and down it as if it were a vertical rod passing through the bar. Hence the two ends of the bar move respectively along the lines of intersection of a cylinder with two equal and similarly situated spheres, the diameter of the cylinder being the length of the bar, and the radius of each sphere the length of either cord.

The proof, then, may be given concisely as follows:—The point of attachment of the moving bar traces out the intersection of a cylinder with a sphere, this—in practical cases—is very approximately that of a cylinder with an inclined plane; the steepness in this case is as the sine of the angle through which the bar is turned. This is directly as the magnitude of the force tending to turn it.

The ordinary trigonometrical treatment may be found in Wiedemann's Galvanismus, B. II., Th. I., s. 289.

## Abstract of one of Euler's papers.

By J. S. Mackay, M.A.

The paper is entitled *Solutio facilis problematum quorumdam geometricorum difficillimorum*, and is printed in *Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tom. xi., pp. 103-123. The volume is for the year 1765; the title-page is dated 1767.

The investigation is concerned with a plane triangle and its four points the orthocentre, the centroid, the inscribed centre, and the circumscribed centre.

Not to complicate a single figure with too many lines, four figures are exhibited. In fig. 40, AM, BN, CP are the perpendiculars from A, B, C on the opposite sides, and E is the orthocentre. In fig. 41, A$a$, B$b$, C$c$ are the medians from A, B, C, and F is the centroid; FQ and CP are perpendicular to AB. In fig. 42, A$\alpha$, B$\beta$, C$\gamma$ are the bisectors of angles A, B, C, and G is the inscribed centre; GR is perpendicular to AB. In fig. 43, S, T, V are the middle points of AB, BC, CA, and SH, TH, VH respectively perpendicular to those sides meet in H the circumscribed centre; AM, CP are perpendicular to BC, AB, and AH is joined.

The sides of the triangle BC, CA, AB are denoted respectively by $a$, $b$, $c$, [see the notation for triangle FGH on p. 54] and the area by A.

### Orthocentre E.

By application of Euclid II. 13, and from the similar triangles ABM, AEP it is found that

$$\mathrm{AP} = \frac{c^2 + b^2 - a^2}{2c};\ \mathrm{PE} = \frac{(c^2 + b^2 - a^2)(a^2 + c^2 - b^2)}{8c\mathrm{A}}.$$

### Centroid F.

From the similar triangles F*c*Q, C*c*P and the preceding result it is found that $\mathrm{AQ} = \frac{3c^2 + b^2 - a^2}{6c}$, $\mathrm{QF} = \frac{2\mathrm{A}}{3c}$.

### Inscribed Centre G.

Since R is the point of contact of the inscribed circle with AB it is found that $\mathrm{AR} = \frac{c + b - a}{2}$, $\mathrm{RG} = \frac{2\mathrm{A}}{a + b + c}$.

### Circumscribed Centre H.

From the similar triangles AHS, ACM it is found that

$$\mathrm{AS} = \tfrac{1}{2}c,\ \mathrm{SH} = \frac{c(a^2 + b^2 - c^2)}{8\mathrm{A}}.$$

Hence when these four points are taken on the same figure, the expressions for their mutual distances will be

$$\mathrm{EF}^2 = (\mathrm{AP} - \mathrm{AQ})^2 + (\mathrm{PE} - \mathrm{QF})^2$$
$$\mathrm{EG}^2 = (\mathrm{AP} - \mathrm{AR})^2 + (\mathrm{PE} - \mathrm{RG})^2$$
$$\mathrm{EH}^2 = (\mathrm{AP} - \mathrm{AS})^2 + (\mathrm{PE} - \mathrm{SH})^2$$
$$\mathrm{FG}^2 = (\mathrm{AQ} - \mathrm{AR})^2 + (\mathrm{QF} - \mathrm{RG})^2$$
$$\mathrm{FH}^2 = (\mathrm{AQ} - \mathrm{AS})^2 + (\mathrm{QF} - \mathrm{SH})^2$$
$$\mathrm{GH}^2 = (\mathrm{AR} - \mathrm{AS})^2 + (\mathrm{RG} - \mathrm{SH})^2.$$

Now here it is particularly to be observed that those distances between our four points ought necessarily to be so expressed that the three sides of the triangle may enter equally into the expressions, since no precedence can be attributed to any one side over the rest in respect of these distances. Accordingly let

$$a + b + c = p,\ ab + ac + bc = q,\ abc = r;\ \text{whence}$$
$$a^2 + b^2 + c^2 = p^2 - 2q,\ a^2b^2 + a^2c^2 + b^2c^2 = q^2 - 2pr,$$
$$a^4 + b^4 + c^4 = p^4 - 4p^2q + 2q^2 + 4pr,$$
$$\mathrm{A}^2 = \tfrac{1}{16}p(-p^8 + 4pq - 3r).$$

Hence, by substitution and after considerable calculation, the expressions for the six distances become

$$EF^2 = \frac{r^2}{4A^2} - \frac{4}{9}(p^2 - 2q)$$

$$EG^2 = \frac{r^2}{4A^2} - p^2 + 3q - \frac{4r}{p}$$

$$EH^2 = \frac{9r^2}{16A^2} - (p^2 - 2q)$$

$$FG^2 = -\frac{1}{9}p^2 + \frac{5}{9}q - \frac{2r}{p}$$

$$FH^2 = \frac{r^2}{16A^2} - \frac{1}{9}(p^2 - 2q)$$

$$GH^2 = \frac{r^2}{16A^2} - \frac{r}{p}.$$

Here it is evident that $EH = \frac{3}{2}EF$, and $FH = \frac{1}{2}EF$, and thus the point H is determined by the points E, F. That is, if the three points E, F, G form a triangle EFG, then the fourth point H will be situated in EF produced so that $FH = \frac{1}{2}EF$, and therefore $EH = \frac{3}{2}EF$.
Hence is deduced $4GH^2 + 2EG^2 = 3EF^2 + 6FG^2$.
[Euler gives no hint of how he obtains this last result. It is got from a theorem which he afterwards (*Acta Academiae Scientiarum Imperialis Petropolitanae, pro Anno* 1780. *Pars Prior*, pp. 92-93) proved as a lemma in order to inscribe in a circle a triangle whose three sides pass through three given points. The theorem is Prop. II. of Matthew Stewart's *Some General Theorems of considerable use in the higher parts of Mathematics*, (Edinburgh, 1746) and may be thus enunciated:—If three collinear points A, C, B be joined to any point D, then $DA^2 BC + DB^2 \cdot AC - DC^2 \cdot AB = AB \cdot AC \cdot BC$.]

To give greater simplicity to these formulae, assume

$$P = p^2,\ Q = \frac{r}{p},\ R = \frac{r^2}{4A^2},$$

so that P, Q, R are quantities involving two dimensions.
Thence are obtained

$$p = \sqrt{P},\ q = \frac{1}{4}P + 2Q + \frac{Q^2}{R},\ r = Q\sqrt{P},$$

and the following expressions:—

$$EF^2 = R - \frac{2}{9}P + \frac{16}{9}Q + \frac{8Q^2}{9R}$$

$$EG^2 = R - \frac{1}{4}P + 2Q + \frac{3Q^2}{R}$$

$$\begin{aligned}
EH^2 &= \frac{9}{4}R - \frac{1}{2}P + 4Q + \frac{2Q^2}{R}\\
FG^2 &= \quad + \frac{1}{36}P - \frac{8}{9}Q + \frac{5Q^2}{9R}\\
FH^2 &= \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2Q^2}{9R}\\
GH^2 &= \frac{1}{4}R - Q.
\end{aligned}$$

Accordingly since of these four points three (unless these three E, F, and H are taken) contain the determination of the fourth, there results the single problem :—

These four points, E, F, G, H, connected with any triangle being given in position, to construct the triangle. (See fig. 44).

Assume $GH = f$, $FH = g$, and $FG = h$,

whence $EF = 2g$, $EH = 3g$, $EG = \sqrt{6g^2 + 3h^2 - 2f^2}$,

and the three following equations are obtained

$$\begin{aligned}
f^2 &= \frac{1}{4}R - Q\\
g^2 &= \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2Q^2}{9R}\\
h^2 &= \frac{1}{36}P - \frac{8}{9}Q + \frac{5Q^2}{9R}.
\end{aligned}$$

From these the values of P, Q, R, and consequently of $p$, $q$, $r$ may be found. The sides of the triangle then will be the three roots of the cubic equation

$$z^3 - pz^2 + qz - r = 0.$$

In the case when these four points are in a straight line (see fig. 45) EG becomes $= 2g - h$, and $g = f - h$.

When substitution is made it appears that the cubic equation for determining the three sides of the triangle has two equal roots, and that consequently the triangle is isosceles.

This case is easy to resolve independently, since the straight line in which the given points are must necessarily cut the triangle into two similar parts, and accordingly the triangle is isosceles. Starting with the assumption that $b = a$, the following values are obtained

$$A = \tfrac{1}{4}c\sqrt{4a^2 - c^2},\ AP = AQ = AR = AS = \tfrac{1}{2}c,$$

$$PE = \frac{c^3}{8A},\ QF = \frac{2A}{3c},\ RG = \frac{2A}{2a + c},\ SH = \frac{c(2a^2 - c^2)}{8A}.$$

If O be the middle point of the base, P, Q, R, S coincide at O, and the six distances between the four points are

$$\mathrm{OF}-\mathrm{OE}=\frac{2(a^2-c^2)}{3\sqrt{4a^2-c^2}} \qquad \mathrm{OG}-\mathrm{OE}=\frac{c(a-c)}{\sqrt{4a^2-c^2}}$$

$$\mathrm{OH}-\mathrm{OE}=\frac{a^2-c^2}{\sqrt{4a^2-c^2}} \qquad \mathrm{OF}-\mathrm{OG}=\frac{(a-c)(2a-c)}{3\sqrt{4a^2-c^2}}$$

$$\mathrm{OH}-\mathrm{OF}=\frac{a^2-c^2}{3\sqrt{4a^2-c^2}} \qquad \mathrm{OH}-\mathrm{OG}=\frac{a(a-c)}{\sqrt{4a^2-c^2}}.$$

Here two cases are to be considered, when $a>c$ and when $a<c$; for if $a=c$, the triangle is equilateral, and the four points coalesce in one.

If $a>c$, the points will be situated as in fig. 46 where HF = ⅓EH, or EF = ⅔EH, and EG < ½EH.

In this case O falls in HE produced beyond E so that $\mathrm{OE}=\dfrac{c^2}{2\sqrt{4a^2-c^2}}$.

If $a<c$, the points will be situated as in fig. 47 where HF = ⅓EH or EF = ⅔EH, but EG > ½EH.

In this case O falls in EH produced beyond H so that $\mathrm{HO}=\dfrac{2a^2-c^2}{2\sqrt{4a^2-c^2}}$; whence if $2a^2<c^2$, the point O falls between H and E.

## *Sixth Meeting, April 8th,* 1886.

DR FERGUSON, F.R.S.E., President, in the Chair.

## On the Divisibility of certain Numbers.

By J. S. MACKAY, M.A.

Of the following properties the first two are obvious; the third was communicated to the Royal Society of London by the Rev. Dr James Booth* in 1854; the others, obtained many years ago, have not as far as I know been remarked. They seem to be more curious than useful.

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* See *Proceedings of the Royal Society of London*, (London, 1856) vol. vii. pp. 42-43.

Any number consisting of

1. One figure repeated is divisible by 11.
2. Two figures „ „ „ „ 101.
3. Three „ „ „ „ „ 7, 11, 13.
4. Four „ „ „ „ „ 73, 137.
5. Five „ „ „ „ „ 11, 9091.
6. Six „ „ „ „ „ 101, 9901.
7. Seven „ „ „ „ „ 11, 909091.
8. Eight „ „ „ „ „ 17, 5882353.
9. Nine „ „ „ „ „ 7, 11, 13, 19, 52579.

The proofs of these properties are very simple. Take for example the third.

The smallest number consisting of three figures repeated is 001001, and all the others are multiples of it.

But $1001 = 7 \times 11 \times 13$.

Dr Booth's proof is:

A number N of six places may be thus written:

$N = 100000a + 10000b + 1000c + 100d + 10e + f$,

which, when divided by 7, will give a quotient $q$, and a remainder $5a + 4b + 6c + 2d + 3e + f$.

Now if $d = a$, $e = b$, $f = c$, this remainder may be written $7(a + b + c)$, which is divisible by 7, whatever be the values of $a$, $b$, $c$.

Similarly for the divisors 11 and 13.

On consulting Dr Booth's paper the other day I find that he states 17 to be a divisor of any number consisting of eight figures repeated. He does not appear to have observed that the other divisor 5882353 is a prime.

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# Projective Geometry of the Sphere.

By R. E. ALLARDICE, M.A.

[*Abstract.*]

If A and B be two fixed points on a great circular arc and P a variable point on the arc, there are two and only two possible positions of the point P corresponding to a given value of the ratio sinAP/sinBP, provided arcs measured in one direction from A or B be considered positive, and in the opposite direction negative; and these two points are antipodal.

In the case of the determination of the position of a ray of a pencil of great circular arcs by means of the corresponding ratio, a complete circular arc takes the place of the two antipodal points in the previous case. This theorem is the polar of the last.

Most theorems relating to transversals in plane geometry may be transferred to the sphere by conical projection from the centre of the sphere on the spherical surface. This was illustrated in the paper in the case of Menelaus's theorem. The same method may be applied to prove that the anharmonic ratio of any transversal of a pencil of four rays is the same; or this may be proved more simply still by means of the theory of projective ranges, or directly in a manner analogous to that employed in plane geometry, by means of the following theorem.

If $h$ be the perpendicular from the vertex A on the base BC of a spherical triangle, then $\sin a \sin h = \sin b \sin c \sin A$.

It may be noticed that the anharmonic ratios of a range and of the polar pencil are the same.

It is obvious that such theorems as that which gives the properties of the complete quadrilateral follow at once for the sphere.

## Application to Sphero-conics.

A sphero-conic is the intersection of a quadric cone with a sphere which has its centre at the vertex of the cone.

A complete sphero-conic consists of two closed curves, antipodal to one another, each of which may be considered a spherical ellipse; or a great circle, drawn through the centre of each perpendicular to one of the axes may be considered as dividing the complete conic into two spherical hyperbolas.

The chief properties of these curves may be deduced at once from the theory of projective pencils. This method gives in succession

Pascal's and Brianchon's theorems;

The properties of the inscribed and of the circumscribed quadrilateral;

The theory of pole and polar by means of the complete quadrilateral;

The properties of conjugate diameters (which however do not bisect their ordinates);

The properties of the foci by means of the theory of involution.

[The principal properties of sphero-conics, a number of which were enunciated in the paper, may be found in Chasles's *Memoir on Cones and Spherical Conics* translated by Rev. Charles Graves.]

## Transformation of Figures.

Similar Figures. According to the ordinary notion of similar figures in plane geometry, such cannot exist on the surface of the sphere, since the area of a figure is determined by its curvature. The following is, however, analogous to the method of transformation by similarity in plane geometry.

If T be a fixed point, P a point on a given curve, Q a point such $\tan\frac{1}{2}TQ/\tan\frac{1}{2}TP$ is constant, the locus of Q is said to be a curve similar to the locus of P.

The curve similar to a circle is another circle.

The tangents at corresponding points on two similar curves are equally inclined to the common radius vector.

The polar of the above method of transformation gives a second method, which is applicable in plane geometry.

Inversion. If in the method of transformation given above the product $\tan\frac{1}{2}TQ\tan\frac{1}{2}TP$ be substituted for the quotient $\tan\frac{1}{2}TQ/\tan\frac{1}{2}TP$, the curve so derived may be called the inverse of the given curve.

In the case of the sphere however the methods of transformation by similarity and by inversion are really not distinct.

To the fact that in transformation by similarity or by inversion tangents at corresponding points are equally inclined to the radius vector, corresponds in the polar methods of transformation the fact that corresponding tangents are equal; and to the fact that the angle of intersection of two curves is unaltered in the one method of transformation, the fact that the distance between the points of contact of common tangents is unaltered in the second method.

[Some of the foregoing results are given in Mulcahy's *Modern Geometry.*]

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## Statical proofs of some Geometrical Theorems.

By John Alison, M.A.

1. Figure 54. If ABC be a triangle, P any point, then the system of forces PA, PB, PC is equivalent to the system PH, PK, PL, where H, K, L are the middle points of BC, CA, AB.

For the resultant of PA, PB is 2 PL,
of PB, PC is 2 PH,
of PC, PA is 2 PK;
therefore the system PA, PB, PC is equivalent to the system PH, PK, PL.

If H′, K′, L′ be the middle points of the sides of the triangle HKL, then PA, PB, PC is equivalent to PH′, PK′, PL′. If the bisection of the sides of the triangles be continued, the points H, K, L approach one another, and finally coincide at a point G, so that the system is equivalent to 3PG. If PN be the resultant of PA, PB, PC, then P, G, N are collinear, and PN = 3PG.

2. If P be at H, the middle point of BC, then the resultant of HA, HB, HC, since HB, HC are in equilibrium, is HA and is equal to 3HG. Hence the three medians are concurrent at G and are divided at G in the ratio 1 : 2.

Also, since HA, HB, HC is equivalent to HH, HK, HL, therefore HA is the resultant of HK, HL. Hence, HKAL is a parallelogram; the line joining the middle points of two sides of a triangle, is parallel to and equal to half of the third side.

3. When P is at S, the centre of the circumscribing circle ABC, then the system SA, SB, SC = SA, ST, where ST is equal to 2SH, the resultant of SB, SC; and the resultant of SA, ST is SO, which is equal to 3SG. O is on the perpendicular from A on BC, and intercepts a part AO equal to 2SH. Hence the perpendiculars from B and C also pass through O, and BO = 2SK, CO = 2SL.

O, the orthocentre, G, the centroid, and S, the circumscribed centre of any triangle are collinear, and OG = 2GS.

S is the orthocentre of the triangle HKL, and G its centroid; S′, its circumscribed centre is found by cutting off GS′ from GO equal to half of SG, that is, S′, the centre of the nine-point circle of any triangle bisects the line joining its orthocentre and circumscribed centre.

4. Figure 55. If A, B, C, D be the vertices of a tetrahedron, then, as in § 1, it may be shown that the system of forces PA, PB, PC, PD is equivalent to 4PG, where G is the centre of mean position of the four points. Let $G_1$, $G_2$, $G_3$, $G_4$ be the centroids of the triangles BCD,

CDA, DAB, ABC; then if P be at $G_1$, the forces PB, PC, PD are in equilibrium, and the resultant of the system is $G_1A$, and is equal to $4G_1G$. Hence the lines $AG_1$, $BG_2$, $CG_3$, $DG_4$ are concurrent at G, and are divided at G in the ratio $1:3$. The general proposition is— If each of $n$ points in space be joined to the centre of mean position of the other $n-1$ points, these lines shall be concurrent at the centre of mean position of the $n$ points, and shall be divided in the ratio $1:n-1$.

5. If P be at H, the middle point of the edge BC of the tetrahedron, then HB, HC are in equilibrium, and the resultant of the system is the resultant of HA, HD, that is, 2HF, where F is the middle point of the opposite edge AD. But the resultant is 4HG, therefore G bisects the line HF. Hence the lines joining the middle points of opposite edges of a tetrahedron are concurrent and bisect each other.

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## *Seventh Meeting, May 14th,* 1886.

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DR FERGUSON, F.R.S.E., President, in the Chair.

## Note on Euclid II. 11.

By the Right Hon. HUGH C. E. CHILDERS.

[The following method of dividing a straight line in medial section was communicated to Mr Munn of the Edinburgh High School, by the Right Hon. Hugh C. E. Childers in January last.]

Let AB be the line. (Fig 56).

Draw $AC = \frac{1}{2}AB$, and at right angles to it. Join CB. Bisect the angle ACB with CD, cutting AB at D. Draw DE at right angles to CB. Then the triangles CAD, CED are equal, and $AD = DE$. Draw a circle with DA and DE radii, cutting AB at F.

Then since DEC is a right angle, BC is a tangent to the circle, and triangles BCA, BDE are similar.

But $BA = 2CA$; therefore $BE = 2DE = AF$.

$$\begin{aligned}\text{Then } BA \cdot BF &= BE^2,\\ &= AF^2,\end{aligned}$$

and F is the required point of division.

# On the Theory ot Contours, and its Applications in Physical Science.

By W. Peddie.

## PART II.

1. In the first part of this paper we have considered merely the contours of curves, that is, contour points, and the method of obtaining the various physical diagrams. In this part we shall consider chiefly the contours of surfaces; that is, contour lines.

If any curve be cut by planes parallel to that of $(x, y)$ and if the various points of intersection be projected on any one of these planes, say $z = 0$, the contour points so obtained will evidently lie on a definite line, and the line will be more accurately indicated in proportion as the number of intersecting planes is greater and their mutual distance is less. It will be given without any break in continuity by projecting every point of the curve upon the plane $z = 0$. But such a line may be regarded as the intersection, by the plane $z = 0$, (see fig. 48) of a cylindrical surface whose generating lines are parallel to the $z$-axis and are drawn from the given curve to meet that plane. We have here then the intersection of a given surface by a surface over which $z$ is constant. But this satisfies our definition of a contour line. This case of a cylindrical surface supplies the simplest system of contour lines by giving $z$ different values. The contours are all superposed in the diagram, but are not in general conterminous. The only case in which they would be conterminous is that in which the same values of the $x$ and $y$ co-ordinates of a point on the curve correspond to different values of the $z$-co-ordinate.

2. In the case of a non-cylindrical surface, no part of the contours will be superposable in general. The contours of a sphere, for example, are concentric circles. And, just as in the case of contour points the steepness of slope of the original curve is indicated by the closeness of the contour points on the $x$-axis for equal increments of $y$, so in the case of contour lines the steepness of slope of the surface is indicated by the closeness of the contour lines for equal increments of $z$ (see fig. 49). The contours are closer when their radii are large.

Again, the contours of a right circular cone are also concentric circles. These circles however are all at equal distances apart for equal increments of $z$.

In my paper on the Representation of the Physical Properties of substances by means of Surfaces, read before this Society two sessions ago, I have collected together the methods of deducing the general properties of a surface from its contours. In this connection Maxwell's and Cayley's investigations referred to in that paper should be consulted. The differences between Maxwell's diagram of the contours of an inland basin and that of the contours of an insular highland, as mapped in our Proceedings for 1883-84, should be noted.

3. We shall now consider special surfaces the contours of which may be used to indicate certain physical properties.

We have the equation

$$g = 4\pi^2 l/t^2$$

connecting $l$ the length of a pendulum, $t$ its time of oscillation, and $g$ the value of gravity. If $t^2$ and $l$ be measured along two rectangular horizontal axes, and $g$ be measured vertically from the same origin, we obtain a surface the contours of which give the values of $t^2$ and $l$ for any given value of $g$. This surface is obviously that which Maxwell has called a *skew screw surface* (fig. 50), and its contours are straight lines through the origin variously inclined to the axes.

The intrinsic equation of the circle is

$$s = a\phi$$

where $a$ is the radius, and $\phi$ is the angle between the radius-vector and the initial line. Hence the intrinsic equation of one involute is

$$s' = \frac{a}{2}\phi^2.$$

This involute is the one which meets the circle at the position from which $\phi$ is reckoned, and $s'$ is measured along it from this point. If we consider $a$ to be the mass of a moving body, and $\phi$ to be its speed, $s$ and $s'$ are respectively its momentum and kinetic energy. Fig. 51 represents two circles with their involutes satisfying the above conditions. The curves in that figure may be regarded as the contours of a right circular cone and an associated surface so formed that its intersection by any plane parallel to that of the diagram is an involute of the circle in which the cone is cut by the same plane.

The acceleration and speed of a body falling under the action of gravity, and the space passed over by it, are given by the known equations

$$a = g,$$

Hence these various quantities can be represented also by the contours of the surfaces just considered. Only in this case $s$ and $s'$ must not be measured from the point for which $\phi$ is zero.

Again, in the case of a thermo-electric circuit composed of two dissimilar metals, the electromotive force, E, is given in terms of $t$, the difference of temperature of the two junctions, by means of the formula

$$E = a + bt + ct^2.$$

Also the thermo-electric power $e$ is given by the equation

$$e = \frac{dE}{dt} = b + 2ct.$$

Hence the electromotive force and thermo-electric power are representable by means of the same surfaces.

4. As an additional example the contours of the surface showing the relation of the pressure, volume, and temperature of a substance may be taken. These are shown for water-substance in our Proceedings for 1883-84. The contours are isothermals, when pressure and volume are the co-ordinate quantities in the diagram. It is only when the temperature considered happens to be one corresponding to a contour in the diagram that the relation of pressure and volume can be found. This fact, that only a finite number of curves corresponding to different values of the quantity which is constant along each curve, but varies from one to another, can be mapped, constitutes the great defect of the method of contours. But it can be entirely got rid of by using trilinear co-ordinates, and in addition the variation of a fourth quantity can be shown. For a perfect gas we have the equation

$$pv = ct$$

where $p$, $v$, and $t$ represent respectively the pressure, volume, and temperature, and $c$ is a quantity which depends on the nature of the gas and varies from one to another. Fig. 52 shows the contours for different values of $c$ the triangle being equilateral for convenience. Temperature is measured by the distance from the vertical side of the triangle of reference, and pressure and volume from the inclined sides. The equation shows that the curves are hyperbolas with vertical and horizontal axes. No part of the hyperbolas outside the triangle of reference has any physical meaning, as then either the pressure, volume, or temperature, or any two, or all of them, would be negative. This is impossible of course in a perfect gas.

It is evident from this figure that pressure, volume, and temperature for any one gas are continuously represented.

5. Figure 52 evidently represents the contours of a surface by planes parallel to the plane of the diagram which may be looked upon as that corresponding to zero value of $c$. When $c$ is zero, the hyperbola becomes the straight lines coinciding with the sides AB, AC of the triangle of reference. When $c$ is infinite, the side BC is part of the corresponding hyperbola. All points of vertical lines through B and C lie upon the surface. These lines bound the parts of one sheet of the surface which correspond to real physical states of the gas, from those which do not. Outside the triangle the surface is evidently an overhanging precipice.

If $c$ and $t$ have each a definite value, the point on the contour is undetermined. Let P (fig. 53) be the point giving the proper ratios of $p$, $v$, and $t$. Draw PM, PN, parallel to the sides of the triangle of reference. Since the asymptotes of the hyperbola are parallel to the sides AB, AC, it follows that that part of the tangent at P intercepted by the sides of the triangle is bisected at the point of contact. Therefore $\mathrm{AM} = \mathrm{MQ}$, and $\mathrm{AN} = \mathrm{NR}$. Now the compressibility $k$ of a gas is given by the ratio $dv/vdp$. But $dv/dp = \mathrm{MQ}/\mathrm{MP} = \mathrm{NP}/\mathrm{MP} = v/p$ where $v$ and $p$ are, as formerly stated, the perpendiculars from P on the sides AB, AC respectively. Hence $k = 1/p$, that is, the compressibility of a perfect gas is the reciprocal of the pressure, which is a known result. Similarly, it can be shown that the expansibility is inversely proportional to the absolute temperature.

The work done during isothermal expansion can also be found from the diagram. The position of the point P gives the ratio of $p$, $v$, and $t$, but since $t$ has a known constant value, the actual values of $p$ and $v$ are also known. Hence PN $(= p \operatorname{cosec} \mathrm{BAC})$ is a known known function of $v$. If P move to P′ the area $\mathrm{PNN'P'} = \int \mathrm{PN}\,dv$ $= \operatorname{cosec} \mathrm{BAC} \int p\,dv$ is a given multiple of the work do[illegible]

6. The applicability of the method of contour[illegible] [illegible]hysical problems is evident. Electric stream lines and [illegible] flo[illegible] may be regarded as contours of a surface. An[illegible] of equipotential lines crossing unit length of a stream [illegible] to indicate the strength of current. So also air-c[illegible] isobars, isothermals and flow-lines of heat, &c., are recta[illegible] of contours.

# On certain forms of Vibration.

By CHARLES CHREE, M.A.

Certain cases of motion in a gas contained in a spherical or in a cylindrical envelope or surrounding a sphere or a cylinder have been considered by Professor Stokes and Lord Rayleigh, of which a clear idea may be obtained from chapters xvii. and xviii. of Rayleigh's *Sound*. Their chief object is to determine the motion in the gas when the motion in the bounding surface is given, or to determine the modification in the nature of a wave being propagated through the gas by the presence of an obstacle.

Now if the spherical or cylindrical boundary be looked on as an elastic solid, then any vibration instituted in a contiguous liquid or gas will tend to produce vibrations in the boundary, precisely as any vibration set up in the solid will tend to produce vibrations in the fluid.

The object of the present paper is to consider certain simple forms of vibration when the solid boundary and the fluid are treated as one system.

If $u$ denote the radial displacement in a sphere or spherical shell at a distance $r$ from the centre, $\rho$ being the density, and $m$, $n$ in Thomson and Tait's notation the elastic constants for the material, the equation for radial vibrations as given in Lamè's *Leçons sur L' Elasticité* * becomes

$$\frac{d\delta}{dr}=\frac{\rho}{m+n}\frac{d^2u}{dt^2} \qquad \ldots \qquad \ldots \qquad (1)$$

It must be noticed that $n\equiv\mu$, $m\equiv\lambda+\mu$ and $\delta\equiv\theta$ in Lamè's notation. Also the dilatation $\delta=\frac{du}{dr}+\frac{2u}{r} \qquad \ldots \qquad \ldots \qquad (2)$, as may be seen from a previous paper by the author in the *Proceedings*, noticing that $e=\frac{du}{dr}, f=g=\frac{u}{r}$.

The surface condition may be put in the form

$$(m+n)\delta-4n\frac{u}{r}=\text{normal traction} \qquad \ldots \qquad (3).†$$

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[illegible]0.

In (1) assume $u \propto \cos kt$, and let $\frac{\rho}{m+n} = a^2$ ... ... (4),

then we have to determine $u$ as a function of $r$ from

$$\frac{d}{dr}\left(\frac{du}{dr} + \frac{2u}{r}\right) + k^2a^2u = 0.$$

This may be put in the form

$$\frac{d^2}{dr^2}\left(ur^{\frac{1}{2}}\right) + \frac{1}{r}\,\frac{d}{dr}\left(ur^{\frac{1}{2}}\right) + ur^{\frac{1}{2}}\left\{k^2a^2 - \left(\frac{3}{2}\right)^2\right\} = 0 \quad \ldots \quad (5).$$

This is the well-known Bessel's equation,* and since $\frac{3}{2}$ is not an integer a complete solution is, A′ and B′ being arbitrary constants,

$$ur^{\frac{1}{2}} = \mathrm{A}'\mathrm{J}_{\frac{3}{2}}(kar) + \mathrm{B}'\mathrm{J}_{-\frac{3}{2}}(kar) \quad \ldots \quad (6).$$

If the sphere be solid B′ must be zero, but for a shell both functions may exist. Leaving out a constant multiplier, we have†

$$\left.\begin{aligned} \mathrm{J}_{\frac{3}{2}}(z) &= \frac{1}{\sqrt{z}}\left(\frac{\sin z}{z} - \cos z\right) \\ \mathrm{J}_{-\frac{3}{2}}(z) &= -\frac{1}{\sqrt{z}}\left(\frac{\cos z}{z} + \sin z\right) \end{aligned}\right\} \quad \ldots \quad (7).$$

For convenience let $\mathrm{A}'(ka)^2 = \mathrm{A}$, and $\mathrm{B}'(ka)^2 = \mathrm{B}$, then for a solid sphere we have

$$u = \frac{\mathrm{A}}{(ka)^2}\cos kt \frac{1}{r}\left\{\frac{\sin kar}{kar} - \cos kar\right\} \quad \ldots \quad (8);$$

whence $$\delta = \mathrm{A}\cos kt \frac{\sin kar}{kar} \quad \ldots \quad \ldots \quad (9).$$

From (3) the periods $\frac{2\pi}{k}$ of the free vibrations of the sphere are given by $(m+n)\delta - 4n\frac{u}{r} = 0$ when $r = a$, if $a$ denote the radius of the sphere.

Hence we find

$$kaa \cot kaa = 1 - \frac{m+n}{4n}(kaa)^2 \quad \ldots \quad (10).$$

One root is $kaa = 0$, none of the others are very small.

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* See the *Bessel'schen Functionen* of Neumann or Lommel.

† Lommel, p. 118. Also Lord Rayleigh's *Sound*, vol. i., p. 278, Eqn. (3).

Now suppose an incompressible fluid to surround the sphere and to be otherwise unlimited. Then the vibrations existing in the sphere will cause vibrations of the same pitch in the fluid of a purely radial character also. Thus the velocity potential $\phi$ for the fluid motion is a function only of $r$ and $t$, therefore $\nabla^2\phi=0$* reduces to

$$\frac{d^2\phi}{dr^2}+\frac{2}{r}\,\frac{d\phi}{dr}=0 \qquad \dots \qquad \dots \qquad (11).$$

Obviously a solution is $\phi=\left(\frac{\mathrm{C}}{r}+\mathrm{D}\right)\sin kt \qquad \dots \qquad \dots \qquad (12),$

where C and D are arbitrary constants. This solution gives the velocity $\frac{d\phi}{dr}$ zero at infinity, and from the equation at the surface of the sphere we can determine C and D in terms of A and so satisfy all necessary conditions.

It is obvious that the fluid in contact with the sphere must have the same normal velocity as the sphere; thus when $r=a$,

$$\frac{d\phi}{dr}=\frac{du}{dt} \qquad \dots \qquad \dots \qquad (13).$$

Again the pressure on the surface of the sphere is the fluid pressure, and thus the variable parts of these must be identical. Now, if $\sigma$ be the density and $p$ the variable part of the fluid pressure, and if the action of gravity be disregarded, and the square of the velocity neglected,

$$\frac{d\phi}{dt}+\frac{p}{\sigma}=0 \qquad \dots \qquad \dots \qquad (14)\dagger;$$

therefore from (3), when $r=a$

$$\sigma\frac{d\phi}{dt}=(m+n)\delta-4n\frac{u}{r} \qquad \dots \qquad \dots \qquad (15).$$

Determining C and D in terms of A from (13) and (15), and substituting in (12) we obtain

$$\phi=\frac{\mathrm{A}}{k}\sin kt\left[-\left(1-\frac{a}{r}\right)\frac{1}{a^2}\left\{\frac{\sin kaa}{kaa}-\cos kaa\right\}+\frac{1}{\sigma}\left\{(m+n)\frac{\sin kaa}{kaa}\right.\right.$$
$$\left.\left.-\frac{4n}{(kaa)^2}\left(\frac{\sin kaa}{kaa}-\cos kaa\right)\right\}\right] \qquad \dots \qquad (16).$$

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* See Lamb's *Motion of Fluids*, § 43.

† See Lamb's *Motion of Fluids*, Eqn. (4), p. 20, putting F $(t)=0$.

This gives $$\frac{d\phi}{dr} = -\frac{Aa\sin kt}{ka^2r^2}\left\{\frac{\sin kaa}{kaa} - \cos kaa\right\} \quad \ldots \quad (17).$$

Thus the velocity in the fluid is independent of $\sigma$, and of course varies inversely as the square of the distance from the centre.

If the vibration be one of those natural to the sphere, we see from (10) that (16) becomes, the co-efficient of $\frac{1}{\sigma}$ vanishing,

$$\phi = -\frac{A}{ka^2}\sin kt\left(1 - \frac{a}{r}\right)\left\{\frac{\sin kaa}{kaa} - \cos kaa\right\} \quad \ldots \quad (18).$$

In this case we see from (14) that the variable part of the fluid pressure is directly proportional to its density.

If $kaa$ be small, which however excludes the vibrations natural to the sphere, an approximate solution is from (17)

$$\frac{d\phi}{dr} = -\tfrac{1}{3}Ak\frac{a^3}{r^2}\sin kt \quad \ldots \quad \ldots \quad (19).$$

Suppose next that the solid sphere is surrounded by a gas. Then (8) and (9) will still represent the motion in the sphere, and if $\sigma$ now denote the density of the gas, whose vibrations are supposed not very large, (13) and (15)* will also give the surface conditions. But if $\phi$ denote the velocity potential in the gas, the vibrations in which are purely radial, the equation is now†

$$c^2\left(\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr}\right) = \frac{d^2\phi}{dt^2} \quad \ldots \quad \ldots \quad (20),$$

where $c$ is the velocity of sound in the gas.

Assume $\phi \propto \sin kt$, then (20) may be put in the form

$$\frac{d^2}{dr^2}\left(r^{\frac{1}{2}}\phi\right) + \frac{1}{r}\frac{d}{dr}\left(r^{\frac{1}{2}}\phi\right) + \left\{\frac{k^2}{c^2} - \frac{(\frac{1}{2})^2}{r^2}\right\}(r^{\frac{1}{2}}\phi) = 0 \qquad (21).$$

The two solutions of this equation are $J_{\frac{1}{2}}\left(k\frac{r}{c}\right)$ and $J_{-\frac{1}{2}}\left(k\frac{r}{c}\right)$, both of which will apply since the gas is excluded from the origin.

From Lommel, or by direct trial, $J_{\frac{1}{2}}(z) \propto \frac{1}{\sqrt{z}}\sin z$, and $J_{-\frac{1}{2}}(z) \propto \frac{1}{\sqrt{z}}\cos z$; therefore a complete solution of (21) is

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* See Lord Rayleigh's *Sound*, Eqn. (6) of § 244.

† Lamb's *Motion of Fluids*, Eqn. (12) of § 167.

$$\phi = \sin kt \frac{\mathrm{E}\sin\left(\frac{kr}{c}\right) + \mathrm{F}\cos\frac{kr}{c}}{\frac{kr}{c}} \quad \ldots \quad (22),$$

where E and F are arbitrary constants.

If for shortness

$$\mathrm{H} \equiv \frac{1}{ka^2}\left\{\frac{\sin kaa}{kaa} - \cos kaa\right\} \quad \ldots \quad \ldots \quad (23),$$

$$\text{and } \mathrm{K} \equiv \frac{1}{k\sigma}\left[(m+n)\frac{\sin kaa}{kaa} - \frac{4n}{k^2a^2a^2}\left\{\frac{\sin kaa}{kaa} - \cos kaa\right\}\right] \quad (24),$$

then it will be found from (13) and (15) that

$$\mathrm{E} = \mathrm{A}\left[-\mathrm{H}\cos\frac{ka}{c} + \mathrm{K}\left\{\cos\frac{ka}{c} + \frac{ka}{c}\sin\frac{ka}{c}\right\}\right]$$

and

$$\mathrm{F} = \mathrm{A}\left[\mathrm{H}\sin\frac{ka}{c} - \mathrm{K}\left\{\sin\frac{ka}{c} - \frac{ka}{c}\cos\frac{ka}{c}\right\}\right].$$

Whence from (22)

$$\phi = \mathrm{A}\sin kt \frac{(\mathrm{K}-\mathrm{H})\sin\left(k\frac{r-a}{c}\right) + \mathrm{K}\frac{ka}{c}\cos\left(k\frac{r-a}{c}\right)}{\frac{kr}{c}} \quad (25).$$

If the vibration be one of those natural to the sphere, then, from (10), K = 0, and reducing H we get

$$\phi = -\mathrm{A}ka^2\sin kt\frac{m+n}{4n}\,\frac{\sin kaa}{kaa}\,\frac{\sin\left(k\frac{r-a}{c}\right)}{\frac{kr}{c}} \quad (26).$$

From (14) it appears that in this case the variable part of the pressure is everywhere and at all times directly proportional to $\sigma$.

If the vibration be not one of those natural to the sphere, but be such that $kaa$ is small, then approximately

$$\mathrm{H} = \tfrac{1}{3}ka^2 \quad \ldots \quad (27), \qquad \text{and } \mathrm{K} = \frac{3m-n}{3k\sigma} \quad \ldots \quad (28).$$

Introducing these values in (25) and noticing that $\frac{ka}{c}$, though for large values of $r$ not $\frac{kr}{c}$, is also small we get a very simple result.

To give some idea of the relative magnitudes of $c$ and $\frac{1}{a}$ we may mention that in the C. G. S. system of units $c$ is about 33000 for air at 0°C.

It is difficult to give exact values for $a$ as different experimenters get different results, in many cases doubtless owing to the difference between specimens nominally the same; but for metals such as brass or iron it is safe to say that to take $\frac{1}{a} = 15000$ gives at all events a correct idea of its dimensions. Thus if we assume $kaa$ small we must also take $\frac{ka}{c}$ small.

From (25) we can determine A so as to get a given amplitude of vibration over any spherical surface concentric with the solid sphere. We might thus suppose the motion to be forced by an arbitrary purely radial vibration of any elastic spherical membrane containing the gas, in whole or in part, and concentric with the solid sphere. The consequent forced vibrations in the sphere are then given by (8).

Suppose next a spherical shell of elastic solid structure, the radii of whose surfaces are $r = a$, and $r = a'$, containing a gas and itself surrounded by gas. We need not in the first place consider the surrounding gas.

From (5), (6), (7), and (8) we see that a suitable solution for a vibration of period $\frac{2\pi}{k}$ in the shell is

$$u = \frac{\cos kt}{k^2a^2r}\left[A\left\{\frac{\sin kar}{kar} - \cos kar\right\} + B\left\{\frac{\cos kar}{kar} + \sin kar\right\}\right] \quad (29),$$

giving

$$\delta = \frac{\cos kt}{kar}\left\{A\sin kar + B\cos kar\right\} \quad \ldots \quad (30).$$

By making the expression (3), namely, $(m+n)\delta - 4n\frac{u}{r}$ vanish over both surfaces $r = a$ and $r = a'$ we find, after eliminating A and B, for the equation giving the frequency of the free radial vibrations of the shell

$$ka(a-a')\cot ka(a-a') \cdot 4n\{(m+n)k^2a^2aa' + 4n\} =$$
$$(m+n)^2k^4a^4a^2a'^2 - 4n(m+n)k^2a^2(a^2+a'^2) + 16n^2(1+k^2a^2aa') \ \ldots \ (31).$$

If $\frac{a-a'}{a}$ be small an approximate solution will be found to be

$$kaa = \frac{2\sqrt{\{n(3m-n)\}}}{m+n} \quad \dots \quad \dots \quad (32).$$

For the contained gas we see from (21) and (22) that a suitable solution is

$$\phi = \mathrm{E}\sin kt \frac{\sin\frac{kr}{c}}{\frac{kr}{c}} \quad \dots \quad \dots \quad \dots \quad (33),$$

as this remains finite and $\frac{d\phi}{dr}$ vanishes when $r = 0$.

The equations (13) and (15) apply at the surface $r = a$. From these, if H and K have the same form as in (23) and (24), and if further

$$\mathrm{L} = \frac{1}{ka}\left\{\frac{\cos kaa}{kaa} + \sin kaa\right\} \quad \dots \quad \dots \quad \dots \quad (34)$$

and

$$\mathrm{M} = \frac{1}{k\sigma}\left[(m+n)\frac{\cos kaa}{kaa} - \frac{4n}{k^2a^2a^2}\left\{\frac{\cos kaa}{kaa} + \sin kaa\right\}\right] \dots \quad (35),$$

we find, noticing that $\mathrm{KL} - \mathrm{HM} = \frac{m+n}{k^3a^3a\sigma}$,

$$\mathrm{A} = \frac{\mathrm{E}k^2a^3\sigma c}{m+n}\left[\mathrm{L}\sin\frac{ka}{c} - \mathrm{M}\left\{\sin\frac{ka}{c} - \frac{ka}{c}\cos\frac{ka}{c}\right\}\right] \quad \dots \quad (36)$$

and

$$\mathrm{B} = \frac{\mathrm{E}k^2a^3\sigma c}{m+n}\left[\mathrm{K}\left\{\sin\frac{ka}{c} - \frac{ka}{c}\cos\frac{ka}{c}\right\} - \mathrm{H}\sin\frac{ka}{c}\right] \quad \dots \quad (37).$$

Substituting these values in (29) we have the solution completely expressed in terms of the one arbitrary constant E, which may be determined so that the vibration at any distance from the centre, whether in the gas or in the shell, may have a given amplitude.

If the vibration be one of those natural to the shell, on the usual theory, we have the right hand side of (15) vanishing, and thus from (33) we should also have $\sigma\mathrm{E}\frac{\sin\frac{ka}{c}}{\frac{ka}{c}} = 0$.

If we regard terms in $\sigma$ as negligible this is satisfied, but otherwise the result is in general impossible. We cannot suppose E to vanish, as then no vibration would exist either in the gas or in the shell. The solution $\frac{ka}{c} = i\pi$ could happen only through accident, as $k$ is one of a series of numbers already determined. If for instance we suppose the shell very thin we see from (32) that $ka = i\pi$ could happen only if $ac = \frac{2\sqrt{\{n(3m-n)\}}}{i\pi(m+n)}$ ... ... ... (38).

The ratio $m : n$ is not known with much certainty for any solids, in fact it forms one of the most disputed points in the range of Physical Science. We will at least approach to an average result if we take Poisson's value 2 for this ratio. This would require $iac = \frac{21}{44}$ roughly, which, even taking $i = 1$, is considerably too small for most combinations of a gas and a solid. The smaller $c$ is, the more nearly would (38) be true; thus carbonic acid gas for which $c$ is about 26000 would more nearly suit the conditions than air, which in turn would suit very much better than hydrogen.

The real solution of the difficulty is that if $\sigma$ be not negligible a vibration natural to the shell, as usually defined, cannot really exist, while if the period approach closely to one of those natural to the shell E must be small if $\sigma$ be big. Thus if the gas be at a high pressure it will very seriously muffle those vibrations of the shell which at small pressures would be far the most important. It is well known that the amplitude of vibration of any elastic body under the influence of a periodic force is comparatively small, unless the period of the force approximate to one of the natural periods of vibration of the elastic body. Thus great importance may attach to the values of $\sigma$ and of $ac$ as modifying the amplitude and period of vibration under certain forms of excitation.

For the gas surrounding the shell we may take the solution (22), writing E′ for E and F′ for F. Then to determine E′ and F′ in terms of A and B, and so of E, we have the equations (13) and (15) applied to the surface $r = a'$. If we denote the value of H &c., when $a'$ is written for $a$ by H′ &c., we find

$$E' \sin\frac{ka'}{c} + F' \cos\frac{ka'}{c} = \frac{ka'}{c}(K'A + M'B) \quad \dots \quad (39)$$

and

$$E' \cos\frac{ka'}{c} - F' \sin\frac{ka'}{c} = (K' - H')A + (M' - L')B \quad (40).$$

These give E′ and F′ simply in terms of A and B, and so of E by means of (36) and (37). Thus the vibrations in the shell as well as in the enclosed and surrounding gases are all expressed in terms of the one constant E. Thus the motion might be produced by forced displacement of the surface of the shell, or of any concentric spherical membrane. We can obviously extend the method to any series of concentric elastic solid spherical shells separated by layers of gas. Exactly the same method also applies to concentric contiguous layers of different elastic solids or of different gases.

If the vibration be one natural to the shell we must have $K'A + M'B = 0$, and therefore from (39) and (40)

$$F' \operatorname{cosec} \frac{ka'}{c} = -E' \sec \frac{ka'}{c} = (H'A + L'B) \quad \ldots \qquad (41).$$

There is no difficulty in satisfying these equations for all values of $\sigma$. It should however be noticed that since $E'$ and $F'$ are determined, a difficulty would arise if the gas surrounding the shell were in turn bounded by a surface such as to restrict the motion in the gas. If for instance this new boundary were a rigid spherical shell this would require $\frac{d\phi}{dr}$ to vanish for a given value of $r$, which of course could happen only accidentally; though if the radius of this boundary were large this defect might be neglected. Further, in the case of the natural vibration, the above equation (41) shews from the values of $H'$ and $L'$, to be obtained from (23) and (36) by writing $a'$ for $a$, that $E'$ and $F'$ do not contain terms in $\frac{1}{\sigma}$. Thus from (22) the value of $\phi$ for the surrounding gas contains no term in $\frac{1}{\sigma}$, and therefore from (14) the variable part of the pressure in this gas is directly proportional to $\sigma$.

Thus by diminishing the density of a gas surrounding a sphere performing free radial vibrations the sound should appear weakened to an observer surrounded by this rarefied atmosphere. If the rarefied gas be surrounded by an elastic solid envelope, the variable gaseous pressure $\sigma \frac{d\phi}{dt}$ on the inside of this envelope being diminished will in turn diminish the variation in the pressure transmitted through the envelope to the gas surrounding it. Thus a lowering in the sound would also be perceived by an observer though not actually inside the space containing the rarefied gas.

Similar methods are applicable to the case of the vibrations in two dimensions of a right circular cylinder containing or surrounded by a fluid. We shall suppose no displacement parallel to the length of the cylinder which we assume to be so long that we may neglect the conditions at its ends. The following solution would thus strictly assume the ends of the cylinder maintained at a constant distance apart by suitably varying normal forces, which suitable supports would give. The results should also be limited to points in the fluid the distances of which from the central cross section of the cylinder are considerably less than half the length of the cylinder. If, however, the cross section be small, these limitations may be disregarded, excluding points in the axis of the cylinder produced.

If $u$ denote the displacement for purely radial vibrations in the cross section, the equation of motion for the cylinder is

$$\frac{d\delta}{dr}=\frac{\rho}{m+n}\frac{d^2u}{dt^2} \quad \dots \quad \dots \quad (42),^*$$

where now, however

$$\delta=\frac{du}{dr}+\frac{u}{r} \quad \dots \quad \dots \quad (43).^*$$

Also, the surface condition is now

$$(m+n)\delta-2n\frac{u}{r}=\text{normal traction} \dots \quad (44).^*$$

Assuming a vibration of period $\frac{2\pi}{k}$ we have to determine $u$ as a function of $r$ from

$$\frac{d^2u}{dr^2}+\frac{1}{r}\frac{du}{dr}+u(k^2a^2-\frac{1}{r^2}) \quad \dots \quad \dots \quad (45),$$

where $a$ has the same meaning as before.

This is a Bessel's equation, the two solutions being denoted by $J_1(kar)$ and $Y_1(kar)$, the latter occurring only for a hollow cylinder.

Thus, A being an arbitrary constant, we have for a solid cylinder

$$u=A\cos kt J_1(kar) \quad \dots \quad \dots \quad (46).$$

If $b$ denote the radius of the cylinder, it will be found from (44) that the periods of free vibration, as usually defined, are given by

$$(m+n)J_1'(kab)+(m-n)\frac{J_1(kab)}{kab}=0 \quad \dots \quad (47)$$

where $J_1'(kab)\equiv\frac{d}{kadr}J_1(kar)$, putting $r=b$ after differentiation.

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[*] See a paper by the author in the *Philosophical Magazine*, February 1886, pp. 81, 82.

Suppose now the cylinder surrounded by an incompressible fluid of density $\sigma$. The velocity potential will be given by

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}=0 \qquad \ldots \qquad \ldots \qquad (48).$$

A suitable solution is

$$\phi=\sin kt(\mathrm{C}\log r+\mathrm{D}) \qquad \ldots \qquad \ldots \qquad (49),$$

where C and D are arbitrary constants.

The surface conditions are obviously

$$\frac{du}{dt}=\frac{d\phi}{dr} \qquad \ldots \qquad \ldots \qquad (50)$$

and

$$(m+n)\delta-2n\frac{u}{r}=\sigma\frac{d\phi}{dt} \qquad \ldots \qquad \ldots \qquad (51)$$

when $r=b$.

From these two equations C and D are given in terms of A, and it will be found that (49) becomes

$$\phi=\mathrm{A}\sin kt\left[-kb\mathrm{J}_1(kab)\log\frac{r}{b}+\frac{a}{\sigma}\left\{(m+n)\mathrm{J}_1'(kab)+(m-n)\frac{\mathrm{J}_1(kab)}{kab}\right\}\right] \qquad \ldots \qquad \ldots \qquad (52)$$

Thus the velocity in the fluid is given by

$$\frac{d\phi}{dr}=-\frac{\mathrm{A}kb}{r}\mathrm{J}_1(kab)\sin kt \qquad \ldots \qquad \ldots \qquad (53),$$

which is independent of $\sigma$, and of course inversely proportional to the distance from the axis.

If the vibration be one of those natural to the cylinder the coefficient of $\frac{a}{\sigma}$ in (52) vanishes. Thus $\sigma\frac{d\phi}{dt}$, that is, the variable pressure in the fluid is directly proportional to $\sigma$.

For a gas surrounding a solid cylinder the velocity potential is given by

$$c^2\left(\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}\right)=\frac{d^2\phi}{dt^2} \qquad \ldots \qquad \ldots \qquad (54),$$

where $c$ denotes the velocity of sound in the gas.

Assuming a vibration of period $\frac{2\pi}{k}$, we get to determine $\phi$ as a function of $r$ the Bessel's equation of zero order, viz.,

$$\frac{d^2\phi}{dr^2}+\frac{1}{r}\frac{d\phi}{dr}+\frac{k^2}{c^2}\phi=0 \quad \ldots \quad \ldots \quad (55).$$

The two solutions are denoted by $J_0\left(\frac{kr}{c}\right)$ and $Y_0\left(\frac{kr}{c}\right)$, the latter occurring only when the gas is excluded from the axis $r=0$. Supposing this the case, we may take

$$\phi=\sin kt\left[CJ_0\left(\frac{kr}{c}\right)+DY_0\left(\frac{kr}{c}\right)\right] \quad \ldots \quad (56),$$

where C and D are arbitrary constants. Determining them from (50) and (51), we get

$$\begin{aligned}\phi=\frac{A}{\Delta}\sin kt\Bigg[&cJ_1(kab)\left\{Y_0\left(\frac{kb}{c}\right)J_0\left(\frac{kr}{c}\right)-J_0\left(\frac{kb}{c}\right)Y_0\left(\frac{kr}{c}\right)\right\}\\ &+\frac{a}{\sigma}\left\{(m+n)J_1'(kab)+(m-n)\frac{J_1(kab)}{kab}\right\}\left\{J_0\left(\frac{kr}{c}\right)Y_0'\left(\frac{kb}{c}\right)\right.\\ &\left.-Y_0\left(\frac{kr}{c}\right)J_0'\left(\frac{kb}{c}\right)\right\}\Bigg] \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (57),\end{aligned}$$

Where for shortness $\Delta\equiv J_0\left(\frac{kb}{c}\right)Y_0'\left(\frac{kb}{c}\right)-Y_0\left(\frac{kb}{c}\right)J_0'\left(\frac{kb}{c}\right)$.

It is easy to prove* that $\Delta=\frac{kb}{c}$ (numerical factor), and so $\Delta$ can never vanish, and no forced vibration could become infinite.

If the vibration be one of those natural to the cylinder, the coefficient of $\frac{a}{\sigma}$ in (57) vanishes, and the variable part of the pressure in the gas is directly proportional to $\sigma$ while the velocity is independent of $\sigma$.

Considering next a hollow cylinder, we get for the radial vibrations

$$u=\cos kt\{AJ_1(kar)+BY_1(kar)\} \quad \ldots \quad \ldots \quad (58).$$

The free vibrations, as usually defined, are given by the equation in $k$ which results from the elimination of A and B by means of the conditions $(m+n)\delta-2n\frac{u}{r}=0$ at both surfaces $r=b$ and $r=b'$. If $\frac{b'-b}{b}$

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* See a paper by the author in the *Messenger of Mathematics*, June 1885, p. 20.

be very small an approximate solution will be found to be

$$kab = \frac{2\sqrt{(mn)}}{m+n} \quad \ldots \quad \ldots \quad (59).$$

For a gas inside this cylinder the velocity potential will be given by the single term

$$\phi = \mathrm{C}\sin kt \mathrm{J}_0\left(\frac{kr}{c}\right) \quad \ldots \quad \ldots \quad (60).$$

The equations (50) and (51) at the surface $r=b$ give

$$\frac{\mathrm{C}}{c}\mathrm{J}_0'\left(\frac{kb}{c}\right) = -\left\{\mathrm{A}\mathrm{J}_1(kab) + \mathrm{B}\mathrm{Y}_1(kab)\right\} \quad \ldots \quad (61),$$

and $$\sigma \mathrm{C}\mathrm{J}_0\left(\frac{kb}{c}\right) = \mathrm{A}a\left\{(m+n)\mathrm{J}_1'(kab) + (m-n)\frac{\mathrm{J}_1(kab)}{kab}\right\}$$
$$+ \mathrm{B}a\left\{(m+n)\mathrm{Y}_1'(kab) + (m-n)\frac{\mathrm{Y}_1(kab)}{kab}\right\} \quad \ldots \quad (62).$$

These determine A and B in terms of C.

If the vibration be one of those natural to the shell the second side of (62) vanishes. If $\sigma$ be not negligible this could happen only if $\mathrm{C}=0$, or if $\mathrm{J}_0\left(\frac{kb}{c}\right)=0$. But the ratio of A : B and the value of $k$ are determined from the conditions that the normal pressures vanish over the surfaces $r=b$ and $r=b'$; therefore $\mathrm{C}=0$ would from (61) require both A and B to vanish, and so no vibration would exist. So again, $\mathrm{J}_0\left(\frac{kb}{c}\right)$ could vanish only through accident. For instance, in a very thin shell (59) shows that this could happen only if $\mathrm{J}_0\left\{\frac{2\sqrt{(mn)}}{ac(m+n)}\right\}=0$. Since the least root of $\mathrm{J}_0(z)$ is about 2.4 there are certainly very few cases in which this could occur. The explanation of the difficulty is exactly the same as for the corresponding case of the spherical shell, and so need not be repeated.

If the shell be surrounded by the same or any other gas, the equations (50) and (51) at the outer surface $r=b'$ suffice to determine the two arbitrary constants which occur in the velocity potential for the gas in terms of C. Thus all the details of the motion are known provided the amplitude of the vibration at any one distance from the axis of the shell be given. The results are so exactly similar to those already obtained for the sphere that it is unnecessary to point out how they may be extended to any number of coaxial cylindrical layers of different substances, solid and fluid. We would only remark that a diminution of density in the gas surrounding the cylinder has exactly the same effect in diminishing the variable part of the pressure, and so lowering the sound, as it had in the case of the spherical shell.

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*Eighth Meeting, June 11th,* 1886.

Dr. Ferguson, F.R.S.E., President, in the Chair.

## A Problem in Combinations.

By Alexander Robertson, M.A.

### I.

Given sets of balls of different colours, in how many ways may they be arranged in line so that no two balls of the same colour shall come together.

If we have two colours only, and the same number $m$ of each colour, there are evidently two arrangements possible; if we have $m$, $m-1$ respectively, only one arrangement is possible; if we have $m$, $m-2$; $m$, $m-3$, &c., no arrangement is possible. We may write these results

$$(m, m) = 2,\ (m, m-1) = 1,\ (m, m-2) = 0,\ (m, m-3) = 0, \text{ \&c.}$$

In the sequel we shall consider three colours only, say $m$ white balls, $n$ black balls, and $p$ red balls. The number of arrangements is denoted by $(mnp)$, and in general we shall take $m$ greater than $n$ and $n$ greater than $p$, but not always.

Keeping $m$ and $n$ constant, the smallest value of $p$ is $m-n-1$. There is also a superior limit to $p$; taking $n$ as then the smallest we should have, smallest value of $n = p - m - 1$, therefore $p = m + n + 1$, and therefore the total number of values of $p$ is $2n+3$. For example, let $m=5$ and $n=2$, the smallest value of $p$ is 2 and the greatest 8; or $p$ may have seven values, so that arrangements are possible with 522, 523, 524, 525, 526, 527, 528. If, however, $m$ and $n$ are equal, the case is slightly different; then the smallest value of $p$ is 0, and the greatest is $2m+1$, or $p$ has $2m+2$ values.

We proceed now to discuss the number of arrangements of 4 white (W), 3 black (B), and 2 red (R) balls, or (432). The natural way would be to write the 9 numerals on 9 cards, and first find the possible arrangements of the 4 white balls, by considering in how many ways 123......9 may be arranged 4 at a time so that no two consecutive numbers shall come together. We find that we have 15 such arrangements, which are given in the table, with the remainders of 5 places for the other balls.

| | | Remainders. | Values. |
|---|---|---|---|
| (1) | 1357 | 24689 | 6 |
| (2) | 1358 | 24679 | 6 |
| (3) | 1359 | 24678 | 3 |
| (4) | 1368 | 24579 | 6 |
| (5) | 1469 | 24578 | 4 |
| (6) | 1379 | 24568 | 3 |
| (7) | 1468 | 23579 | 6 |
| (8) | 1469 | 23578 | 4 |
| (9) | 1479 | 23568 | 4 |
| (10) | 1579 | 23468 | 3 |
| (11) | 2468 | 13579 | 10 |
| (12) | 2469 | 13578 | 6 |
| (13) | 2479 | 13568 | 6 |
| (14) | 2579 | 13468 | 6 |
| (15) | 3579 | 12468 | 6 |
| | | | 79 in all. |

We have now to take these remainders and divide each into groups of 3's and 2's, rejecting all those in which two consecutive numbers come together. Thus, taking the first remainder 24689, it gives

| | |
|---|---|
| 246 | ~~89~~ |
| 248 | 69 |
| 249 | 68 |
| 268 | 49 |
| 269 | 48 |
| ~~289~~ | |
| 468 | 29 |
| 469 | 28 |
| ~~489~~ | |
| ~~689~~ | |

So that rejecting the unsuitable ones, we get six possible arrangements. We may call this the value due to this first remainder. Treating them all in the same way we get the third column of the table, or values due to the different remainders, and the sum of these being 79 gives us the result

$$(432) = 79.$$

Generalizing this method, the number of arrangements of the $m$ W balls is $C_m^{n+p+1}$. The following table gives specimens of the value due to a good many remainders. The value is found to depend entirely on the number and range of the sequences in the different remainders. This will abridge the work very much. The value is a maximum when there is no sequence, and is then $C_n^{n+p}$ as in the 11th remainder in the above table.

| Species of sequences. | Values. | Species of sequences. | Values. |
|---|---|---|---|
| (0) | $C_n^{n+p}$ | (3, 3) | $C_{n-2}^{n+p-4}$ |
| (2) | $2C_{n-1}^{n+p-2}$ | (2, 4) | $4C_{n-3}^{n+p-6}$ |
| (3) | $C_{n-1}^{n+p-2}$ | (2, 5) | $2C_{n-3}^{n+p-6}$ |
| (4) | $2C_{n-2}^{n+p-4}$ | (3, 4) | $2C_{n-3}^{n+p-6}$ |
| (5) | $C_{n-2}^{n+p-4}$ | (2, 2, 2) | $8C_{n-3}^{n+p-6}$ |
| (6) | $2C_{n-3}^{n+p-6}$ | (2, 2, 3) | $4C_{n-3}^{n+p-6}$ |
| (2, 2) | $4C_{n-2}^{n+p-4}$ | (2, 3, 3) | $2C_{n-3}^{n+p-6}$ |
| (2, 3) | $2C_{n-2}^{n+p-4}$ | (2, 2, 2, 2) | $16C_{n-3}^{n+p-6}$ |

The work for (432) may therefore be put down as follows—

| Species. | Number. | Value. | Arrangements. |
|---|---|---|---|
| (0) | 1 | 10 | 10 |
| (2) | 8 | 6 | 48 |
| (3) | 3 | 3 | 9 |
| (22) | 3 | 4 | 12 |
| | — | | — |
| | 15 | | 79 |

Application to (543).

Number of arrangements of the $5 = C_5^8 = 56$.

On analysing the remainders as to their sequences we get the following results—

| Species. | Number. | Value. | Arrangements. |
|---|---|---|---|
| (2) | 6 | 20 | 120 |
| (3) | 10 | 10 | 100 |
| (4) | 4 | 6 | 24 |
| (2, 2) | 20 | 12 | 240 |
| (2, 3) | 12 | 6 | 72 |
| (2, 2, 2) | 4 | 8 | 32 |
| | | | 588 |

## II.

We have now to discuss a general theorem which will enable us to make the number of arrangements in any case depend on similar arrangements with a smaller number of elements.

### General Theorem.

$$(mnp) = (m-1,\ n-1,\ p) + (m-1,\ n,\ p-1) + (m,\ n-1,\ p-1)$$
$$+ 2(m-1,\ n-1,\ p-1)$$

Let $W_{mnp}$, $B_{mnp}$, $R_{mnp}$ denote those of $(mnp)$ which commence with a W, a B, or a R respectively, so that

$$(mnp) = (W + B + R)_{mnp}.$$

If we have one W ball more, and put it on at the beginning of the B and R portions of $(mnp)$ we evidently get the W part of $(m+1, n, p)$ that is

$$W_{m+1,\,n,\,p} = (B+R)_{mnp}$$

Similarly $$W_{mnp} = (B+R)_{\overline{m-1},\,n,\,p}$$

so $$B_{mnp} = (W+R)_{m,\,\overline{n-1},\,p}$$

and $$R_{mnp} = (W+B)_{m,\,n,\,\overline{p-1}}$$

therefore $$\begin{aligned}(mnp) &= (W+B+R)_{mnp}\\ &= (B+R)_{m-1,\,n,\,p} + (W+R)_{m,\,n-1,\,p} + (W+B)_{m,\,n,\,p-1}\end{aligned}$$

But $$\begin{aligned}B_{\overline{m-1},\,n,\,p} &= (W+R)_{\overline{m-1},\,\overline{n-1},\,p}\\ R_{\overline{m-1},\,n,\,p} &= (W+B)_{m-1,\,n,\,\overline{p-1}}\\ W_{m,\,\overline{n-1},\,p} &= (B+R)_{\overline{m-1},\,\overline{n-1},\,p}\\ R_{m,\,\overline{n-1},\,p} &= (W+B)_{m,\,\overline{n-1},\,\overline{p-1}}\\ W_{m,\,n,\,\overline{p-1}} &= (B+R)_{\overline{m-1},\,n,\,\overline{p-1}}\\ B_{m,\,n,\,\overline{p-1}} &= (W+R)_{m,\,\overline{n-1},\,\overline{p-1}}\end{aligned}$$

therefore
$$(mnp) = (2W + B + R)_{m,\overline{n-1},\overline{p-1}} + (W + 2B + R)_{\overline{m-1},n,\overline{p-1}}$$
$$+ (W + B + 2R)_{\overline{m-1},\overline{n-1},p}$$
$$= (m, n-1, p-1) + (m-1, n, p-1) + (m-1, n-1, p)$$
$$+ W_{m,n-1,p-1} + B_{m-1,n,p-1} + R_{m-1,n-1,p}$$

But
$$W_{m,n-1,p-1} = (B + R)_{m-1,n-1,p-1}$$
$$B_{m-1,n,p-1} = (W + R)_{m-1,n-1,p-1}$$
$$R_{m-1,n-1,p} = (W + B)_{m-1,n-1,p-1}$$

therefore $W_{m,n-1,p-1} + B_{m-1,n,p-1} + R_{m-1,n-1,p} = 2(m-1, n-1, p-1)$

therefore $(mnp) = (m-1, n-1, p) + (m-1, n, p-1) + (m, n-1, p-1)$
$$+ 2(m-1, n-1, p-1).$$

This gives us a quick method of calculating the number of arrangements in any case.

Application to find (432).

| (211) | (311) | (221) | (321) |
|---|---|---|---|
| (101) = 2 | (201) = 1 | (111) = 6 | (211) = 6 |
| (110) = 2 | (210) = 1 | (120) = 1 | (220) = 2 |
| (200) = 0 | (300) = 0 | (210) = 1 | (310) = 0 |
| 2(100) = 2 | 2(200) = 0 | 2(110) = 4 | 2(210) = 2 |
| 6 | 2 | 12 | 10 |

| (421) | (331) | (322) | (432) |
|---|---|---|---|
| (311) = 2 | (221) = 12 | (212) = 12 | (322) = 38 |
| (320) = 1 | (230) = 1 | (221) = 12 | (331) = 18 |
| (410) = 0 | (320) = 1 | (311) = 2 | (421) = 3 |
| 2(310) = 0 | 2(220) = 4 | 2(211) = 12 | 2(321) = 20 |
| 3 | 18 | 38 | 79 |

By this method the following table of the number of arrangements has been computed—

| | | | |
|---|---|---|---|
| (111) | 6 | (544) | 1668 |
| (211) | 6 | (551) | 30 |
| (221) | 12 | (552) | 222 |
| (222) | 30 | (553) | 1026 |
| (311) | 2 | (554) | 3228 |
| (321) | 10 | (555) | 7188 |
| (322) | 38 | (632) | 10 |
| (331) | 18 | (633) | 100 |
| (332) | 74 | (641) | 5 |
| (333) | 174 | (642) | 70 |
| (421) | 3 | (643) | 445 |
| (422) | 24 | (644) | 1700 |
| (431) | 14 | (651) | 22 |
| (432) | 79 | (652) | 206 |
| (433) | 248 | (653) | 1150 |
| (441) | 24 | (654) | 4315 |
| (442) | 138 | (655) | 11492 |
| (443) | 480 | (661) | 36 |
| (444) | 1092 | (662) | 326 |
| (522) | 6 | (663) | 1882 |
| (531) | 4 | (664) | 7580 |
| (532) | 44 | (665) | 22274 |
| (533) | 212 | (666) | 48852 |
| (541) | 18 | (777) | 339720 |
| (542) | 135 | (888) | 2403588 |
| (543) | 588 | (999) | 17236524 |

Thus, to find (543)

(543)

———

(433) = 248
(442) = 138
(532) = 44
2(432) = 158

———

588

To extend the theorem. Let $\overline{rst}$ denote the arrangement $(m-r,\ n-s,\ p-t)$, then the theorem may be written

$$\begin{aligned}
& & \overline{000} &= \overline{110} + \overline{101} + \overline{011} + 2.\overline{111} \\
&\text{therefore} & \overline{110} &= \overline{220} + \overline{211} + \overline{121} + 2.\overline{221} \\
& & \overline{101} &= \overline{211} + \overline{202} + \overline{112} + 2.\overline{212} \\
& & \overline{011} &= \overline{121} + \overline{112} + \overline{022} + 2.\overline{122} \\
& & 2\overline{111} &= 2(\overline{221}) + 2(\overline{212}) + 2(\overline{122}) + 4(\overline{222}) \\
&\text{therefore} & \overline{000} &= \overline{220} + \overline{202} + \overline{022} \\
& & &\quad + 2(\overline{211} + \overline{121} + \overline{112}) \\
& & &\quad + 4(\overline{221} + \overline{212} + \overline{122}) \\
& & &\quad + 4.\overline{222}
\end{aligned}$$

that is
$$\begin{aligned}
(mnp) &= (m-2,\ n-2,\ p) + (m-2,\ n,\ p-2) + (m,\ n-2,\ p-2) \\
&\quad + 2\{(m-2,\ n-1,\ p-1) + (m-1,\ n-2,\ p-1) \\
&\qquad + (m-1,\ n-1,\ p-2)\} \\
&\quad + 4\{(m-2,\ n-2,\ p-1) + (m-2,\ n-1,\ p-2) \\
&\qquad + (m-1,\ n-2,\ p-2)\} \\
&\quad + 4(m-2,\ n-2,\ p-2)
\end{aligned}$$

and so on to any number of steps.

Another mode of extending it is as follows, and is much simpler.

$$\begin{aligned}
&\text{As before} & \overline{000} &= \overline{110} + \overline{101} + \overline{011} + 2.\overline{111} \\
&\text{therefore} & \overline{111} &= \overline{221} + \overline{212} + \overline{122} + 2.\overline{222} \\
& & \overline{222} &= \overline{332} + \overline{323} + \overline{233} + 2.\overline{333} \\
& & \overline{333} &= \overline{443} + \overline{434} + \overline{344} + 2.\overline{444}
\end{aligned}$$

Add and reject terms which occur on both sides, we get

$$\begin{aligned}
\overline{000} &= \overline{110} + \overline{101} + \overline{011} \\
&\quad + \overline{221} + \overline{212} + \overline{122} \\
&\quad + \overline{332} + \overline{323} + \overline{233} \\
&\quad + \overline{443} + \overline{434} + \overline{344} \\
&\quad + \overline{111} + \overline{222} + \overline{333} + 2.\overline{444}
\end{aligned}$$

whose interpretation is obvious.

## III.

Expression of the number of arrangements by an algebraic formula. Keep $n$ and $p$ constant, say $n = 5$, $p = 2$, and $m$ variable, we have

$(052) = 0$
$(152) = 0$
$(252) = 6$
$(352) = 44$
$(452) = 135$
$(552) = 222$
$(652) = 206$
$(752) = 102$
$(852) = 21$

If we divide by the co-efficients of the 8th power of a binomial we get a regular series.

$$0 \quad 0 \quad \tfrac{3}{14} \quad \tfrac{11}{14} \quad \tfrac{27}{14} \quad \tfrac{111}{28} \quad \tfrac{103}{14} \quad \tfrac{51}{4} \quad 21$$

Multiply by 28, the L.C.M. of the denominators, and difference

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 6 | 22 | 54 | 111 | 206 | 357 | 588 |
| | 0 | 6 | 16 | 32 | 57 | 95 | 151 | 231 |
| | | 6 | 10 | 16 | 25 | 38 | 56 | 80 |
| | | | 4 | 6 | 9 | 13 | 18 | 24 |
| | | | | 2 | 3 | 4 | 5 | 6 |
| | | | | | 1 | 1 | 1 | 1 |

therefore by finite differences

$$(m52) = \frac{6C_2^m + 4C_3^m + 2C_4^m + C_5^m}{28} \times C_m^8$$

It would, perhaps, be simpler to reject the 0, 0 and divide by co-efficients of the 6th power instead of the 8th.

Applying this process to $m11$, $m21$, $m31$, we would get $mn1$
then to $m12$, $m22$, $m32$, „ „ $mn2$
to $m13$, $m23$, $m33$, „ „ $mn3$

and so on.

(We do not actually require to find them all, for $(m21) = (m12)$, &c.)

There results

$$(mn1) = \frac{C_{n-1}^m + 2C_{n-2}^m}{\frac{1}{2}C_3^{n+2}} C_m^{n+2}$$

$$(mn2) = \frac{C_1^n(C_n^m + 2C_{n-1}^m) + 10(2C_{n-2}^m + 3C_{n-3}^m)}{2C_4^{n+3}} \times C_m^{n+3}$$

$$(mn3) = C_2^{n+1}(C_{n+1}^m + 2C_n^m) + 10C_1^{n-1}(2C_{n-1}^m + 3C_{n-2}^m) + 70(3C_{n-3}^m + 4C_{n-4}^m) \times \frac{C_m^{n+4}}{5C^{n+4}}$$

$$\begin{aligned}(mn4) = {} & C_3^{n+2}(C^m_{n+2} + 2C^m_{n+1}) + 10C_2^n(2C^m_n + 3C^m_{n-1}) \\ & + 70C_1^{n-2}(3C^m_{n-2} + 4C^m_{n-3}) + 420(4C^m_{n-4} + 5C^m_{n-5}) \\ & \times \frac{C_m^{n+5}}{10C_6^{n+5}}\end{aligned}$$

$$\begin{aligned}(mn5) = {} & C_4^{n+3}(C^m_{n+3} + 2C^m_{n+2}) + 10C_3^{n+1}(2C^m_{n+1} + 3C^m_n) \\ & + 70C_2^{n-1}(3C^m_{n-1} + 4C^m_{n-2}) + 420C_1^{n-3}(4C^m_{n-3} + 5C^m_{n-4}) \\ & + 2310(5C^m_{n-5} + 6C^m_{n-6}) \\ & \times \frac{C_m^{n+6}}{\frac{35}{2}C_7^{n+6}}\end{aligned}$$

The numerical factors $\frac{1}{2}$, 2, 5, 10 $\frac{35}{2}$, &c., in the denominators are evidently

$$\tfrac{1}{2}C^3_3,\ \tfrac{1}{2}C^4_3,\ \tfrac{1}{2}C^5_3,\ \tfrac{1}{2}C^6_3,\ \tfrac{1}{2}C^7_3,\ \text{\&c.}$$

In the numerators occur the factors 1, 10, 70, 420, 2310, &c., whose law is not so obvious. Denoting them at present by $A_1$, $A_2$, $A_3$, $A_4$, $A_5$, &c., and by comparing the above results we may write

$$\begin{aligned}(mnp) = {} & A_1C_{p-1}^{n+p-2}(C^m_{n+p-2} + 2C^m_{n+p-3}) \\ & + A_2C_{p-2}^{n+p-4}(2C^m_{n+p-4} + 3C^m_{n+p-5}) \\ & + A_3C_{p-3}^{n+p-6}(3C^m_{n+p-6} + 4C^m_{n+p-7}) \\ & \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ & + A_p(pC^m_{n-p} + \overline{p+1}C^m_{n-p-1}) \times \frac{C_m^{n+p+1}}{\frac{1}{2}C_3^{p+2}C_{p+2}^{n+p+2}}\end{aligned}$$

If in this we put $m = n - p$, all the terms vanish except the last,

therefore $(n - p,\ n,\ p) = A_p(pC^{n-p}_{n-1} + \overline{p+1}C^{n-p}_{n-p-1} \times \dfrac{C_{n-p}^{n+p+1}}{\frac{1}{2}C_3^{p+2}C_{p+2}^{n+p+1}}$

In this put $n = p + 1$, we get after some reductions

$$(p + 1,\ p,\ 1) = A_p\frac{(2p + 1)C_{2p+1}^{2p+2}}{\frac{1}{2}C_3^{p+2}C_p^{2p+2}}$$

but we have already seen that

$$\begin{aligned}(p + 1, p, 1) & = \frac{C_{p-1}^{p+1} + 2C_{p-2}^{p+1}}{\frac{1}{2}C_3^{p+2}} \times C_{p+1}^{p+2} \\ & = \frac{C_2^{p+1} + 2C_3^{p+1}(p + 2)}{\frac{1}{2}C_3^{p+2}}\end{aligned}$$

Equate these values and reduce, we get

$$A_p = \frac{p(p + 2)}{12}C_p^{2p+2} = \frac{(2p + 2)!}{12(p - 1)!(p + 1)!}$$

Hence

$$
\begin{aligned}
(mnp) = \frac{1}{12}\Big[ & \frac{4\,!}{2\,!} C_{p-1}^{n+p-2}(C_{n+p-2}^{m} + 2C_{n+p-3}^{m}) \\
& + \frac{6\,!}{3\,!} C_{p-2}^{n+p-4}(2C_{n+p-4}^{m} + 3C_{n+p-5}^{m}) \\
& + \frac{8\,!}{2!4!} C_{p-3}^{n+p-6}(3C_{n+p-6}^{m} + 4C_{n+p-7}^{m}) \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + \frac{(2p+2)\,!}{(p-1)!\,(p+1)!}(pC_{n-p}^{m} + \overline{p+1}C_{n-p-1}^{m})\Big] \\
& \times \frac{C_{m}^{n+p+1}}{\frac{1}{2}C_{3}^{p+2}C_{n-1}^{n+p+1}}
\end{aligned}
$$

By finding the $r^{\text{th}}$ term of this expression, and making some obvious reductions, we get

$$(mnp) = \Sigma \frac{\overline{m-1}r+n+p}{r(r+1)} C_{r-1}^{n-1}C_{r-1}^{p-1}C_{n+p+1-m}^{2r+2}$$

the number of terms is not greater than $p$, but may be less, from the vanishing of the last factor.

This result may also be written more symmetrically

$$(mnp) = \Sigma\left\{\begin{array}{l}(C_{r}^{n}C_{r-1}^{p-1} + C_{r}^{p}C_{r-1}^{n-1})C_{n+p+1-m}^{2r+1} \\ +\,(C_{r}^{n-1}C_{r-1}^{p-1} + C_{r}^{p-1}C_{r-1}^{n-1})C_{n+p-m}^{2r+1}\end{array}\right\}$$

The following method will be found convenient when we wish to calculate all the arrangements possible when two elements $m$, $n$ are kept constant.

Calculate the coefficients K by the formulæ

$$K_{2r} = C_{r}^{n-1}C_{n-r}^{m-1} + C_{r-1}^{n-1}C_{n-r-1}^{m-1}$$

$$K_{2r+1} = 2C_{r}^{n-1}C_{n-r-1}^{m-1}$$

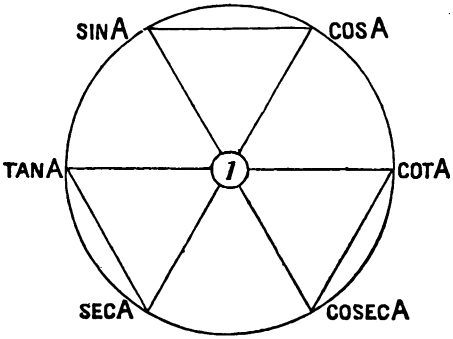
Then

$$
\begin{aligned}
(m,\ n,\ m-n-1) &= K_0 \\
(m,\ n,\ m-n) &= C_{1}^{2n+2}K_0 + K_1 \\
(m,\ n,\ m-n+1) &= C_{2}^{2n+2}K_0 + C_{1}^{2n+1}K_1 + K_2 \\
(m,\ n,\ m-n+2) &= C_{3}^{2n+2}K_0 + C_{2}^{2n+1}K_1 + C_{1}^{2n}K_2 + K_3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(m,\ n,\ m+n-2) &= C_{3}^{2n+2}K_0 + C_{3}^{2n+1}K_1 + C_{3}^{2n}K_2 \ldots\ldots + K_{2n-1} \\
(m,\ n,\ m+n-1) &= C_{2}^{2n+2}K_0 + C_{2}^{2n+1}K_1 + C_{2}^{2n}K_2 \ldots\ldots + C_{2}^{3}K_{2n-1} \\
(m,\ n,\ m+n) &= C_{1}^{2n+2}K_0 + C_{1}^{2n+1}K_1 + C_{1}^{2n}K_2 \ldots\ldots + C_{1}^{3}K_{2n-1} \\
(m,\ n,\ m+n+1) &= K_0 \quad + \quad K_1 + \quad K_2 \ldots\ldots + \quad K_{2n-1}
\end{aligned}
$$

It may be noticed that the sums of alternate terms of these are equal, and that the total sum is always divisible by 16.

## Trigonometrical Mnemonic.

By John Alison, M.A.



If the functions be written as in the above figure, then

1. Of any three functions written consecutively on a circular arc or on a diameter, the mean is the product of the extremes. Thus $\sin A = \tan A \cos A$, $1 = \sin A \operatorname{cosec} A$, and so on.
2. In each triangle the square of the function written at the vertex turned downwards is equal to the sum of the squares of the functions written at the other two vertices. Thus $1^2 = \sin^2 A + \cos^2 A$, and so on.

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## Notes on Integration by parts and by successive reduction.

By George A. Gibson, M.A.

My object in the following notes is to call attention to some points in integration by successive reduction which may be of use in directing the choice of the particular form for the reduced integral in any given case. The remarks apply chiefly to the binomial differential $x^m(a+bx^n)^p dx$ which has been discussed from a different point of view by Dr Muir in vol. iii., p. 100 of the Proceedings.

Let $f(x)$ be a function of $x$ which may be written as the product of two functions $u$, $v$ of $x$. Let $u' = \frac{du}{dx}$, $u_1 = \int u dx$, $v_1 = \int v dx$ &c. Then the integral of $f(x) \cdot dx$ may be expressed in either of the forms

$$\text{(i)}\quad \int f(x)dx = \int u.v dx = u_1.v - \int u_1.v' dx$$

$$\text{(ii)}\quad \int f(x)dx = \int u.v dx = u.v_1 - \int u'.v_1 dx.$$

Now an integral to be found by successive reduction can often be treated by both of these formulae. Take $f(x) = x^m(a+bx^n)^p$ and put $u = x^m$, $v = (a+bx^n)^p$; we have therefore by (i)

$$\int x^m(a+bx^n)^p dx = \frac{x^{m+1}(a+bx^n)^p}{m+1} - \frac{bnp}{m+1}\int x^{m+n}(a+bx^n)^{p-1}dx \quad \text{(A)}$$

Or we may put $u = x^{m-n+1}$, $v = x^{n-1}(a+bx^n)^p$ and we get by (ii)

$$\int x^m(a+bx^n)^p dx = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{bn(p+1)} - \frac{m-n+1}{bn(p+1)}\cdot\int x^{m-n}(a+bx^n)^{p+1}dx \quad \text{(B)}$$

For present purposes, the indices $m$ and $p$ are the most important quantities of which the given integral is a function, and we may therefore denote $\int x^m(a+bx^n)^p dx$ by $\phi(m, p)$. Equation (A) then expresses $\phi(m, n)$ in terms of $\phi(m+n, p-1)$ and (B) in terms of $\phi(m-n, p+1)$; and it is clear that the operation of integrating by parts will generally alter both $m$ and $p$, increasing one and decreasing the other. It should be noted however that $m$ must become $m \pm n$ and $p$, $p \mp 1$. But out of (A) and (B) we can readily get an integral in which only one index is altered, but altered by being *reduced.* For since $x^{m+n} = x^m.\dfrac{(a+bx^n)-a}{b}$ we get

$$\phi(m+n, p-1) = \frac{1}{b}\int x^m(a+bx^n)^p dx - \frac{a}{b}\int x^m(a+bx^n)^{p-1}dx$$
$$= \frac{1}{b}\cdot\phi(m, p) - \frac{a}{b}\cdot\phi(m, p-1).$$

Substituting this value of $\phi(m+n, p-1)$ in (A) we get

$$\phi(m, p) = \frac{x^{m+1}(a+bx^n)^p}{m+np+1} + \frac{anp}{m+np+1}\cdot\phi(m, p-1) \quad \text{(C)}$$

In the same way since $(a+bx^n)^{p+1} = (a+bx^n)^p(a+bx)$, we have
$\phi(m-n, p+1) = a\phi(m-n, p) + b\phi(m, p)$.
Then we get from (B)

$$\phi(m, p) = \frac{x^{m-n+1}(a+bx^n)^{p+1}}{b(m+np+1)} - \frac{a(m-n+1)}{b(m+np+1)}\cdot\phi(m-n, p) \quad \text{(D)}$$

Should it be desired to express $\psi(m, p)$ in terms of an integral in which one of the quantities $m$, $p$ is unaltered while the other is increased, the previous work shows that the integrals we are in search of must be either $\psi(m+n, p)$ or $\psi(m, p+1)$ and it is usually easier to begin with one of these latter forms and work backwards, that is, first break up the factor with increased index and then integrate by parts. Thus we have

$$\psi(m+n, p) = \int x^m \cdot \frac{(a+bx^n)-a}{b} \cdot (a+bx^n)^p dx$$

$$= \frac{1}{b}\int x^m(a+bx^n)^{p+1}dx - \frac{a}{b}\cdot\psi(m, p).$$

Integrate $\int x^m(a+bx^n)^{p+1}dx$ by parts, and we get

$$\psi(m+n, p) = \frac{x^{m+1}(a+bx^n)^{p+1}}{b(m+1)} - \frac{n(p+1)}{m+1}\cdot\phi(m+n, p) - \frac{a}{b}\phi(m, p)$$

$$\text{therefore } \psi(m, p) = \frac{x^{m+1}(a+bx^n)^{p+1}}{a(m+1)} - \frac{b(m+np+n+1)}{a(m+1)}\cdot\phi(m+n, p) \qquad \text{(E)}$$

In the same way by beginning with the form $\phi(m, p+1)$ we get

$$\psi(m, p) = -\frac{x^{m+1}(a+bx^n)^{p+1}}{an(p+1)} + \frac{m+np+n+1}{an(p+1)}\cdot\phi(m, p+1) \qquad \text{(F)}$$

The process applied to $\int x^m(a+bx^n)^p dx$ is equally well suited for the class of integrals of the type $\int\sin^m x\cdot\cos^n x dx$. Thus take $u = \sin^m x\cdot\cos x$ and $v = \cos^{n-1}x$: then we have

$$\int\sin^m x\cdot\cos^n x dx = \frac{\sin^{m+1}x\cdot\cos^{n-1}x}{m+1} + \frac{n-1}{m+1}\int\sin^{m+2}x\cdot\cos^{n-2}x\cdot dx.$$

Denoting $\int\sin^m x\cos^n x dx$ by $\psi(m, n)$ we see that $\psi(m, n)$ may be expressed in terms of $\psi(m+2, n-2)$, $\psi(m-2, n+2)$, $\psi(m, n-2)$, $\psi(m-2, n)$, $\psi(m+2, n)$, $\psi(m, n+2)$. Indeed it would be better for teaching purposes to go through the various reductions for this integral, and then to consider the more general case of the binomial differential.

The results reached may perhaps be expressed as follows, it being understood that the "factors" referred to are of the kind considered in the note:—

In integrating a product of two factors by parts, we get in the first instance a *second integral* in which one of the original factors appears with a lower index and the other with a higher, the decrease and the increase of the indices however following a definite law. This *second integral* may be altered by breaking up the factor whose index has been increased; this breaking up will give the original integral and another in which the index of one factor is the same as in the original integral, while the index of the second factor is lower than at first. Lastly, if a form be desired in which one of the original indices is increased, the other index remaining unaltered, we first observe by how much it is possible to increase the index; we take this integral with increased index, break it up into two integrals, and then integrate by parts.

## Question proposée au Concours Général pour la classe de Mathématiques Spéciales,

### Juin, 1886.

Solution analytique par M. Paul Aubert.

*Étant donnés une surface du second ordre S et deux points A et B, on mène par le point B une sécante qui rencontre la surface aux points C, C′, et le plan polaire du point A au point D. Soient M et M′ les points où la droite AD rencontre les plans qui touchent la surface aux points C et C′. La sécante BD tournant autour du point B, on demande*

1°. *Le lieu décrit par les points M et M′.*

2°. *Ce lieu se compose de deux surfaces du second ordre, dont l' une est indépendante de la position du point B, et l' autre Σ dépend de la position de ce point. Chercher ce que devient la surface Σ quand, dans la construction qui donne les points de cette surface, on fait jouer au point A le rôle du point B, et inversement.*

3°. *Le point A restant fixe, déterminer les positions occupées par le point B quand la surface Σ n' a pas un centre unique à distance finie.*

1°. Rapportons la surface à un tétraèdre de référence ayant pour arêtes opposées la droite AB et la droite polaire conjuguée A′B′.

Les équations de la droite AB sont $\begin{cases} x=0, \\ y=0, \end{cases}$ plan polaire du point A

celles de la droite A′B′ $\begin{cases} z=0 \\ t=0. \end{cases}$

La surface a une équation de la forme

$$l^2x^2+m^2y^2+n^2z^2-p^2t^2=0,$$

où l' un des coefficients $l$, $m$, $n$ peut avoir la forme $\kappa\sqrt{-1}$.

Le point B est défini par trois plans $\begin{cases} z=0 \\ t=0 \\ ax+by=0, \end{cases}$

et la droite mobile BC a pour équations

$$(\mathrm{BC})\begin{cases} ax+by+\nu z=0 \\ \lambda z+\mu t=0, \end{cases}$$

$\lambda$, $\mu$, et $\nu$ étant des paramètres variables.

Le point D est donc défini par le système

$$\begin{cases} x=0 \\ by+\nu z=0 \\ \lambda z+\mu t=0, \end{cases}$$

et par suite la droite AD a pour équations

$$(1)\quad \begin{cases} by+\nu z=0 \\ \lambda z+\mu t=0. \end{cases}$$

On sait qu' un point de la surface peut être déterminé par les quantités $\alpha$, $\beta$, $\gamma$ assujetties aux relations

$$lx=pt\cos\alpha,\ my=pt\cos\beta,\ nz=pt\cos\gamma$$

$$(2)\qquad \cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

Le plan tangent à la surface en ce point a pour équation

$$(3)\qquad lx\cos\alpha+my\cos\beta+nz\cos\gamma-pt=0.$$

Exprimons que le point de contact est sur la droite BC, il vient

$$(4)\quad \begin{cases} \frac{a}{l}\cos\alpha+\frac{b}{m}\cos\beta+\frac{\nu}{n}\cos\gamma=0 \\ \lambda p\cos\gamma+n\mu \qquad =0. \end{cases}$$

Nous aurons le lieu des points M et M′ en éliminant $\lambda$, $\mu$, $\nu$, $\alpha$, $\beta$, $\gamma$ entre les équations (1), (2), (3), et (4).

L' élimination de $\lambda$, $\mu$, $\nu$, et $\gamma$ entre les équations (1) et (4) est immédiate, et il reste à éliminer $\alpha$ et $\beta$ entre les trois équations qui restent :

$$pltx\cos\alpha+pmty\cos\beta=p^2t^2-n^2z^2$$

$$apmtz\cos\alpha+bpltz\cos\beta=blmyz$$

$$p^2t^2\cos^2\alpha+\ p^2t^2\cos^2\beta=p^2t^2-n^2z^2.$$

Cette opération conduit à l' équation

$$[bl(n^2z^2-p^2t^2)+blm^2y^2]^2+[am(p^2t^2-n^2z^2)-bml^2xy]^2$$
$$=(p^2t^2-n^2z^2)[am^2y-bl^2x]^2.$$

En développant cette équation, et faisant passer tous les termes au premier membre, on remarque après quelques réductions la présence du facteur $[m^2y^2+n^2z^2-p^2t^2]$ de sorte que le lieu se compose des deux surfaces

$$m^2y^2+n^2z^2-p^2t^2=0$$

et $$b^2l^2(l^2x^2+m^2y^2)+(b^2l^2+a^2m^2)(n^2z^2-p^2t^2)=0.$$

La première est le cône circonscrit à la surface ayant le point A pour sommet ; la seconde est une surface du second ordre qui admet le tétraèdre de référence comme tétraèdre conjugué.

En écrivant son équation sous la forme

$$b^2l^2(l^2x^2+m^2y^2+n^2z^2-p^2t^2)+a^2m^2(n^2z^2-p^2t^2)=0$$

on voit qu' elle est bitangente à la surface proposée aux points d' intersection de la droite AB avec cette surface.

On pourrait achever complètement la question en revenant aux coordonnées cartésiennes par des transformations simples, mais il peut être intéressant de voir comment on aurait pu arriver au résultat que nous venons d' obtenir en partant des coordonnées cartésiennes. Rapportons donc la surface à un système de trois plans diamétraux conjugués ; elle a pour équation

$$Ax^2+A'y^2+A''z^2-1=0.$$

Soient $x_0\ y_0\ z_0$ les coordonnées du point A,
$x_1\ y_1\ z_1$ „ „ B,
$x\ y\ z$ „ „ M,
$x'\ y'\ z'$ „ „ C,

Un point quelconque de la droite AM a pour coordonnées

$$X=\frac{x+\lambda x_0}{1+\lambda}\quad Y=\frac{y+\lambda y_0}{1+\lambda}\quad Z=\frac{z+\lambda z_0}{1+\lambda}.$$

Nous aurons le point D en assujettissant ces coordonnées à vérifier l' équation du plan polaire du point A

$$AXx_0+A'Yy_0+A''Zz_0-1=0,$$

ce qui donne

$$Ax_0(x+\lambda x_0)+A'y_0(y+\lambda y_0)+A''z_0(z+\lambda z_0)-(1+\lambda)=0.$$

Posons $$Axx_0+A'yy_0+A''zz_0-1=P_0$$
$$Ax_0\ +A'y_0^2+A''z_0^2-1=E_0$$

On a $$\lambda=-\frac{P_0}{E_0}$$

et les coordonnées du point D sont

$$\frac{E_0x - P_0x_0}{E_0 - P_0}, \quad \frac{E_0y - P_0y_0}{E_0 - P_0}, \quad \frac{E_0z - P_0z_0}{E_0 - P_0}.$$

D' autre part, le point D étant sur la droite qui joint le point B au point C, on doit avoir

$$\frac{E_0x - P_0x_0}{E_0 - P_0} = \frac{x_1 + \mu x'}{1 + \mu}$$

et des relations pareilles pour les deux autres coordonnées. Il suffit maintenant d'exprimer que les valeurs de $x'$, $y'$, $z'$ tirées de ces trois relations vérifient les équations

$$(1) \quad Axx' + A'yy' + A''zz' - 1 = 0$$

$$(2) \quad Ax'^2 + A'y'^2 + A''z'^2 - 1 = 0$$

et d' éliminer $\mu$ entre les relations ainsi obtenues pour avoir l' équation du lieu du point $(xyz)$.

En effectuant ce calcul on trouve après l' élimination de $\mu$

$$x' = \frac{[E_0x - P_0x_0]P_1 - [E_0E - P_0^2]x_1}{P_0^2 - E_0E + P_1[E_0 - P_0]}$$

où
$$P_1 = Axx_1 + A'yy_1 + A''zz_1 - 1$$

$$E = Ax^2 + A'y^2 + A''z^2 - 1.$$

$y'$ et $z'$ ont des valeurs de même forme.

Substituant ces expressions dans l' équation (2) en ayant soin de mettre en évidence les facteurs $E_0$, $P_0$,......communs dans les valeurs des trois coordonnées, il vient après quelques réductions simples

$$[E_0^2E - P_0^2][E_1E_0E - P_0^2E_1 - P_1^2E_0 + 2P_1P_0P_0^1] = 0$$

où
$$P_0^1 = Ax_1x_0 + A'y_1y_0 + A''z_1z_0 - 1.$$

Le lieu se compose donc de deux surfaces

$$E_0^2E - P_0^2 = 0$$

qui est le cône circonscrit par le point A, et

$$(3) \qquad E_1E_0E - P_0^2E_1 - P_1^2E_0 + 2P_1P_0P_0^1 = 0$$

On reconnaît l' équation d' une surface du second ordre bitangente à la surface donnée $E = 0$, car les termes qui suivent le premier forment une expression du second degré homogène en $P_1$ et $P_0$, et qui est par conséquent de la forme $(\alpha P_1 + \beta P_0)(\alpha' P_1 + \beta' P_0)$. On voit en même temps que la droite des contacts passe par les points A et B.

2⁰. La réponse à la question 2⁰ résulte immédiatement de la symétrie avec laquelle les indices 0 et 1 entrent dans l' équation (3), et de l'identité $P_0 = P_1^0$. Il suffit en effet pour obtenir ce que devient la surface quand le point B joue le rôle du point A, de permuter dans l' équation (3) les indices 0 et 1. On voit que cette permutation n' amène aucun changement.

3°. Pour résoudre la dernière partie du problème prenons pour axe *oz* le diamètre de la surface qui passe par le point A et pour plan *xoy* le plan diamétral conjugué, les axes *ox* et *oy* étant deux diamètres conjugués de la section. Nous allons chercher les positions occupées par le point B *dans le plan zox* quand la surface Σ n'a pas un centre unique à distance finie.

Dans ces hypothèses on a $x_0 = y_0 = 0$ et $y_1 = 0$.

Si on fait la transformation $Ax = \xi$, $Ay = \eta$, $Az = \zeta$,

l' équation de la surface Σ prend la forme

$$\begin{gathered}(\xi_1^2 + \zeta_1^2 - 1)(\zeta_0^2 - 1)(\xi^2 + \eta^2 + \zeta^2 - 1) - (\zeta\zeta_0 - 1)^2(\xi_1^2 + \zeta_1^2 - 1) \\ - (\zeta_0^2 - 1)(\xi\xi_1 + \zeta\zeta_1 - 1)^2 + 2(\xi\xi_1 + \zeta\zeta_1 - 1)(\zeta\zeta_0 - 1)(\zeta_1\zeta_0 - 1) = 0.\end{gathered}$$

Il suffit de considérer, pour notre objet, les termes du second degré de cette équation ; en les ordonnant il vient

$$\begin{gathered}(\xi_1^2 + \zeta_1^2 - 1)(\zeta_0^2 - 1)\eta^2 + (\zeta_0^2 - 1)(\zeta_1^2 - 1)\xi^2 \\ + [(1 - \zeta_0\zeta_1)^2 - \xi_1^2]\zeta^2 + 2\xi_1(\zeta_0 - \zeta_1)\xi\zeta.\end{gathered}$$

Cette forme quadratique peut se mettre sous la forme d' une somme algébrique de trois carrés ; pour que la surface Σ n' ait pas un centre unique à distance finie il faut et il suffit que le nombre de ces carrés se réduise à deux. Nous aurons donc le lieu des positions occupées par le point B en égalant successivement à zéro chacun des coefficients des carrés. On obtient ainsi immédiatement

$$\begin{gathered}(\xi_1^2 + \zeta_1^2 - 1)(\zeta_0^2 - 1) = 0, \text{ c'est à dire } \xi_1^2 + \zeta_1^2 - 1 = 0 \\ (\zeta_0^2 - 1)(\zeta_1^2 - 1) = 0, \quad ,, \quad ,, \quad ,, \quad \zeta_1^2 - 1 = 0 \\ \xi_1^2(\zeta_0 - \zeta_1)^2 - (\zeta_0^2 - 1)(\zeta_1^2 - 1)[(1 - \zeta_0\zeta_1)^2 - \xi_1^2] = 0.\end{gathered}$$

Celle-ci se réduit à

$$(1 - \zeta_1\zeta_0)^2[\xi_1^2 - (\zeta_1^2 - 1)(\zeta_0^2 - 1)] \quad = 0.$$

Il suffit de remplacer dans ces équations $\xi_1^2$ par $\xi_1^2 + \eta_1^2$ et de revenir aux coordonnées $x$, $y$, $z$ pour avoir les surfaces sur lesquelles doit se trouver le point B.

Soient $b$, $b'$ (fig. 57) les points où *oz* rencontre la surface donnée ; on voit que ces surfaces se composent des plans tangents LM, L'M' aux points où l' axe *oz* rencontre la surface donnée, du plan polaire PQ du point A par rapport à cette surface, et d' une surface du second ordre bitangente à la surface proposée aux points $b$ et $b'$ : cette surface est un hyperboloïde H on un ellipsoïde E suivant que $o$A est supérieur ou inférieur à $ob$. Il est du reste très facile de reconnaître la nature de la surface $\Sigma$ suivant la position du point B sur l' une ou l' autre de ces surfaces en considérant le signe des coefficients des carrés qui ne sont pas nuls et en formant la condition pour que la surface $\Sigma$ soit du genre cylindre.

Fig. 1

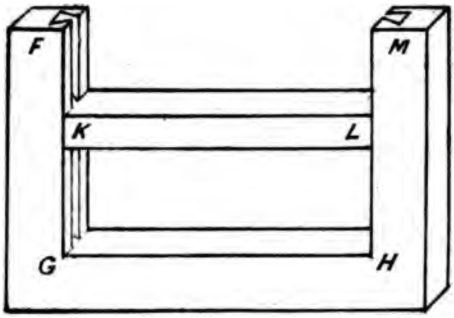


Fig. 2

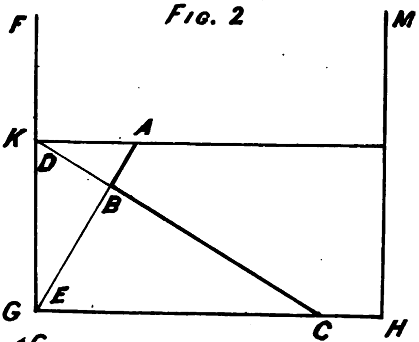


Fig. 3

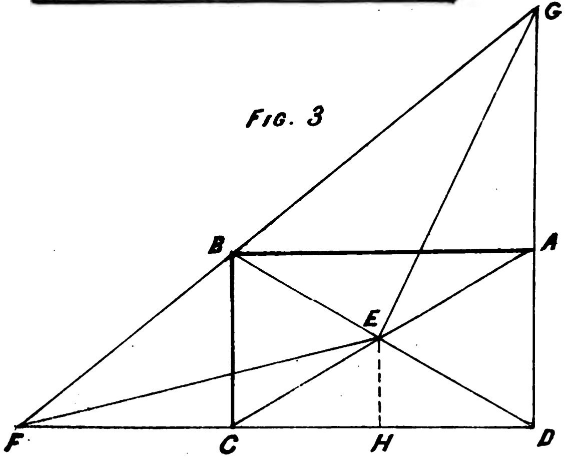


Fig. 4

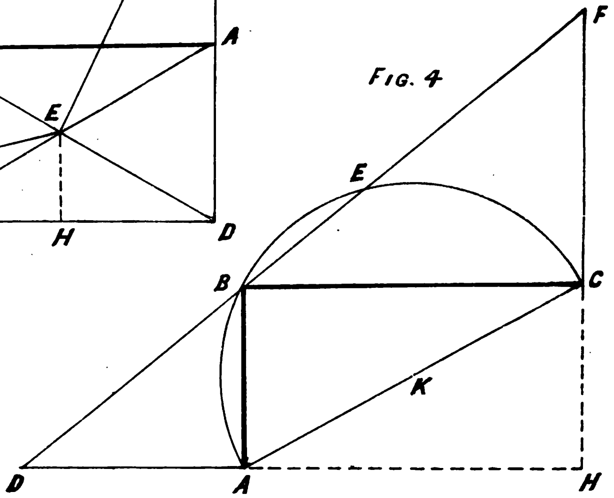
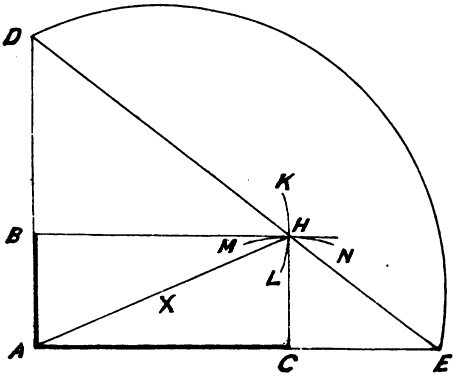


Fig. 5



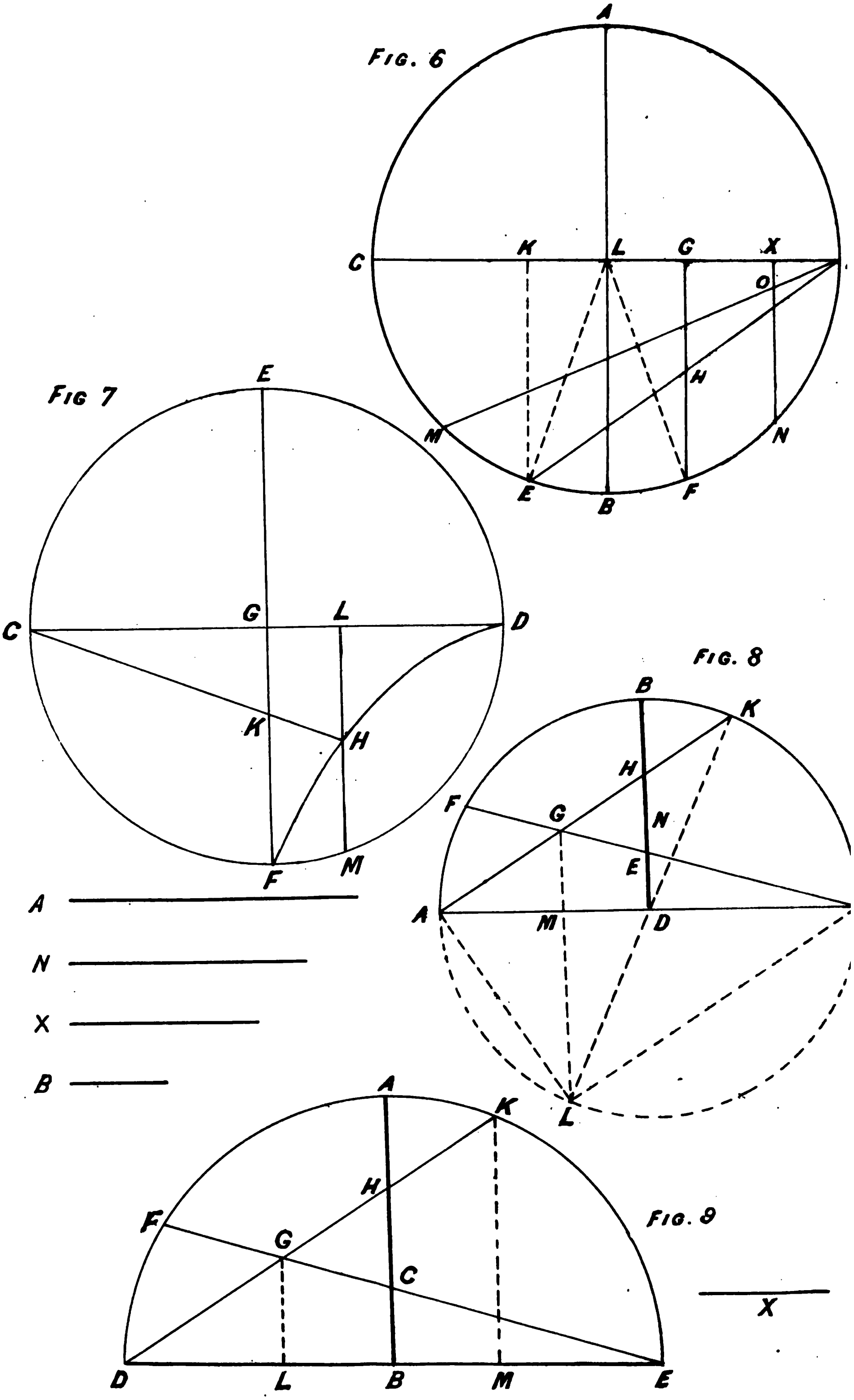


FIG. 6
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FIG. 8
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FIG. 33

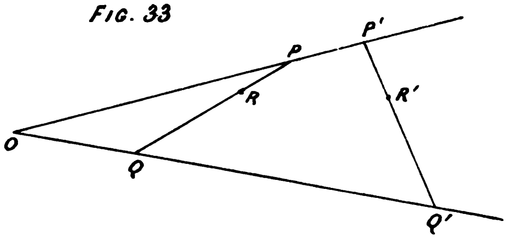


FIG. 34

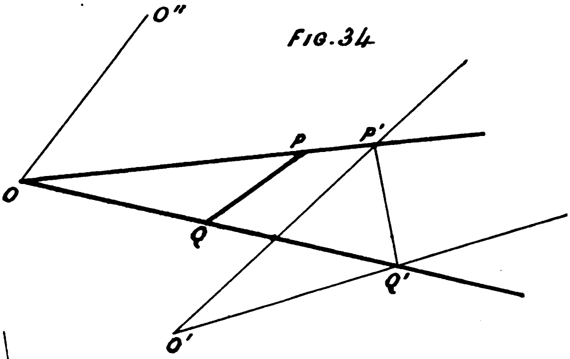


FIG. 35

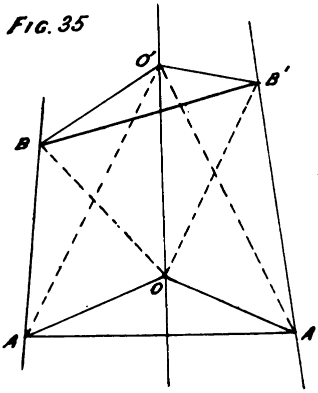


FIG. 36

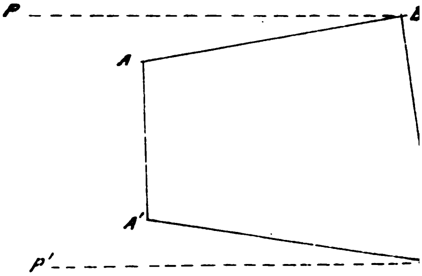


FIG. 10

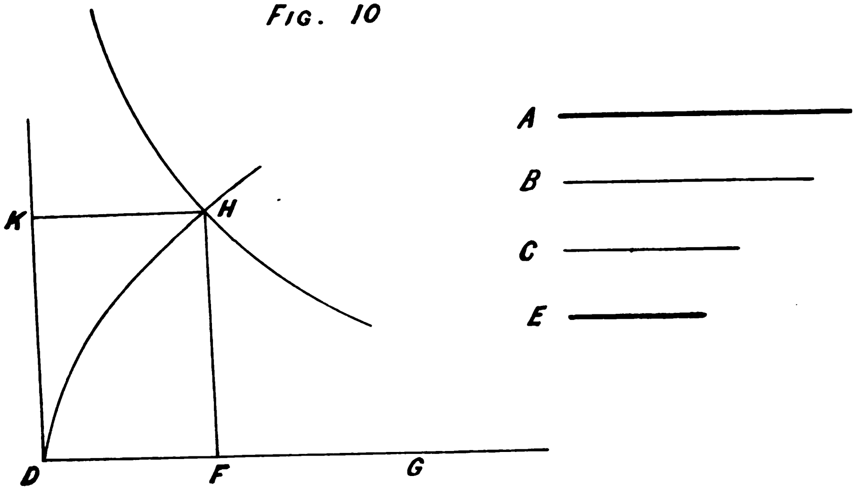


FIG. 11

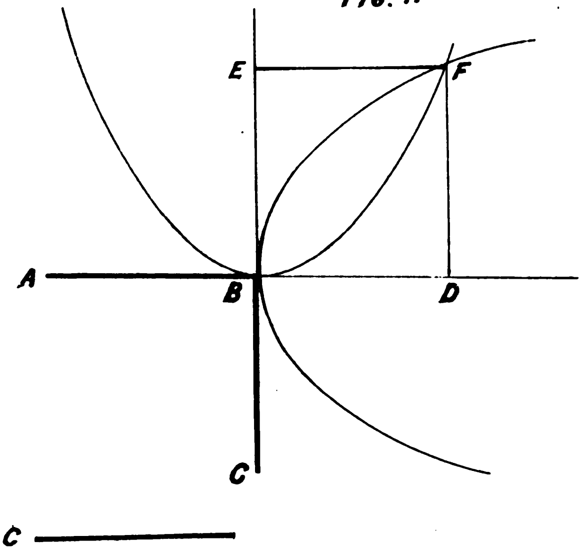


FIG. 12

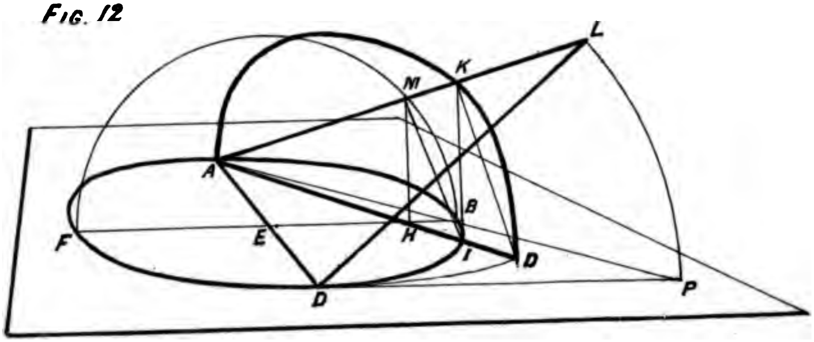


FIG. 10

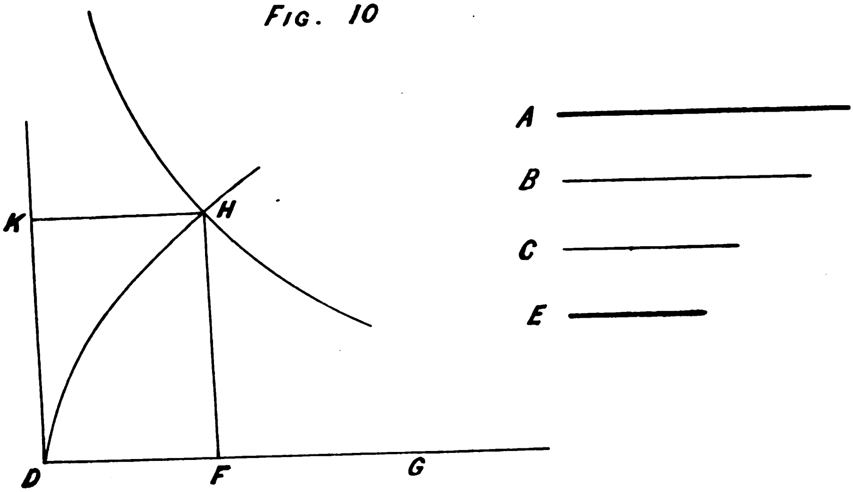


FIG. 11

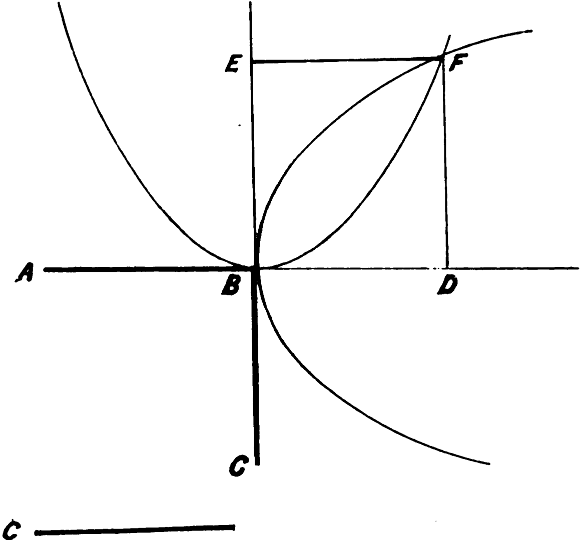


FIG. 12

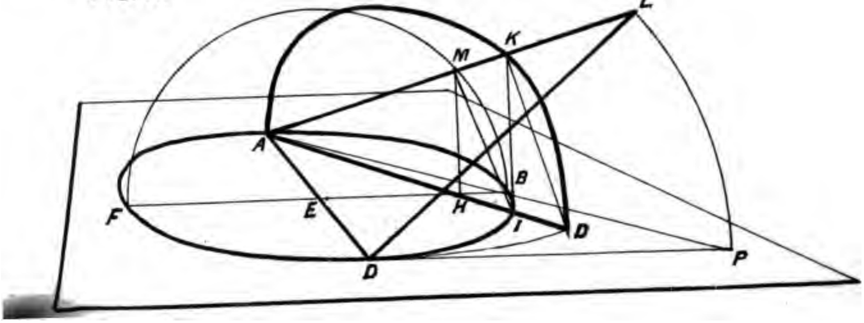


FIG. 13

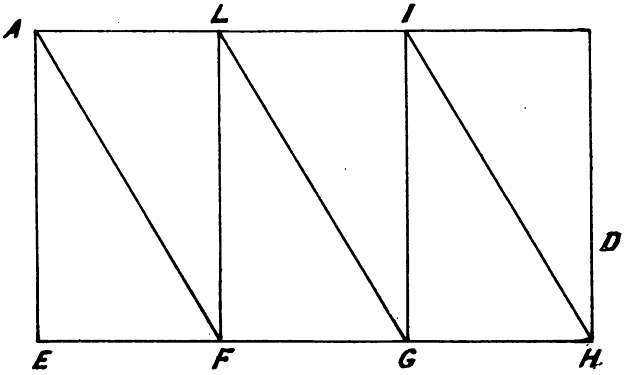
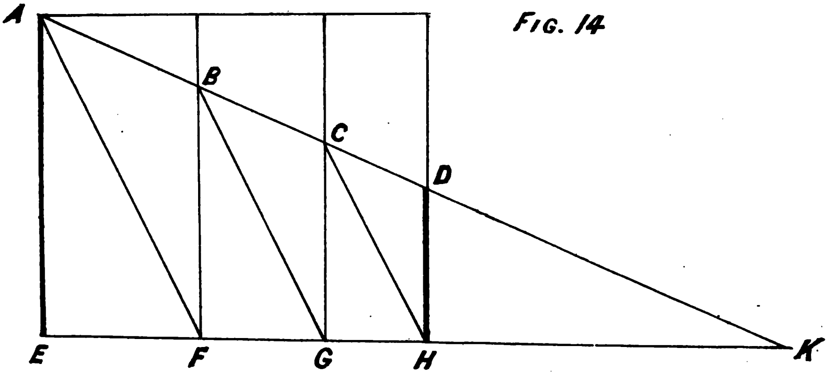


FIG. 14



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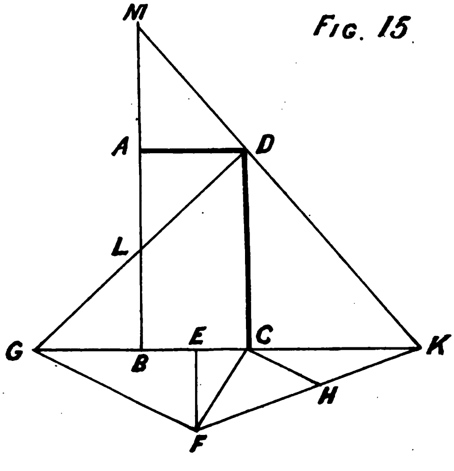


FIG. 48

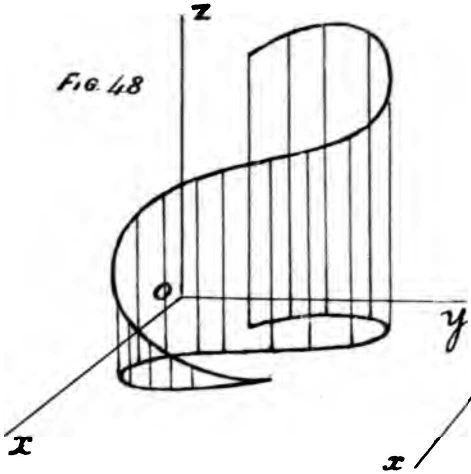


FIG. 49

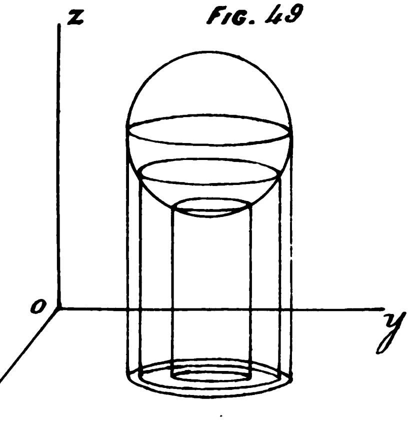


FIG. 50

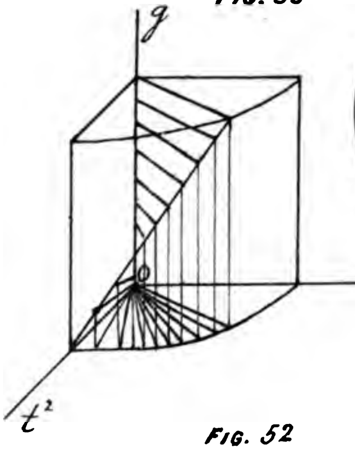
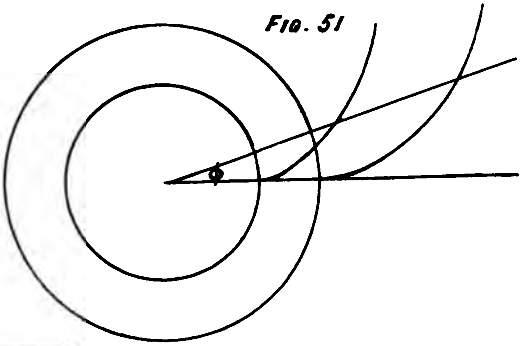


FIG. 51



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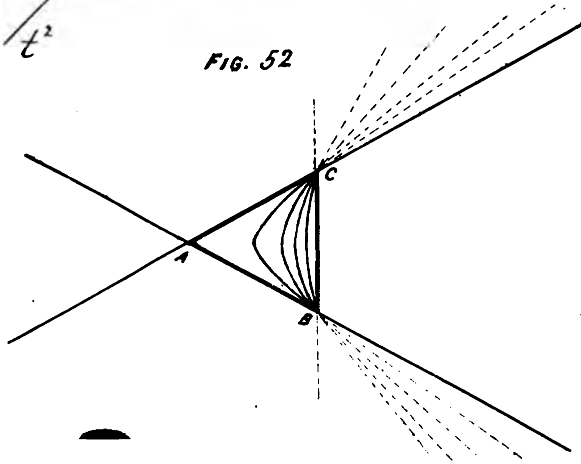

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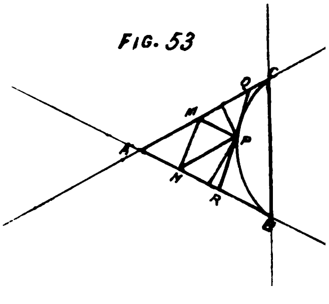

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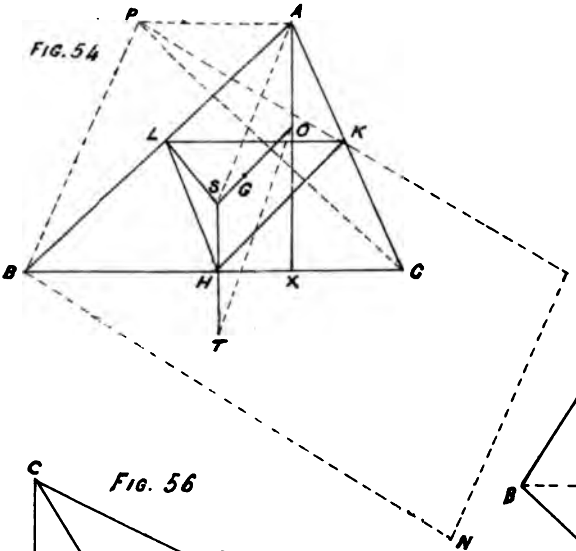

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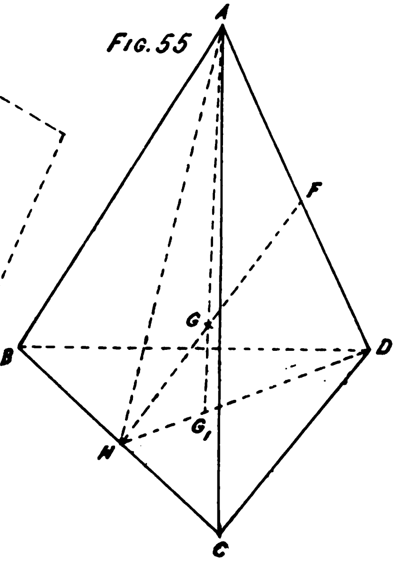


FIG. 56

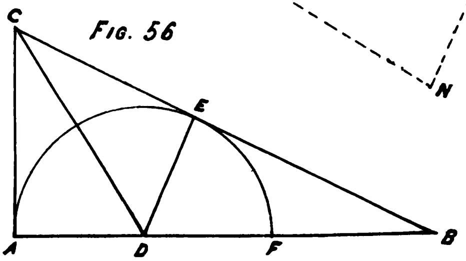


FIG. 57

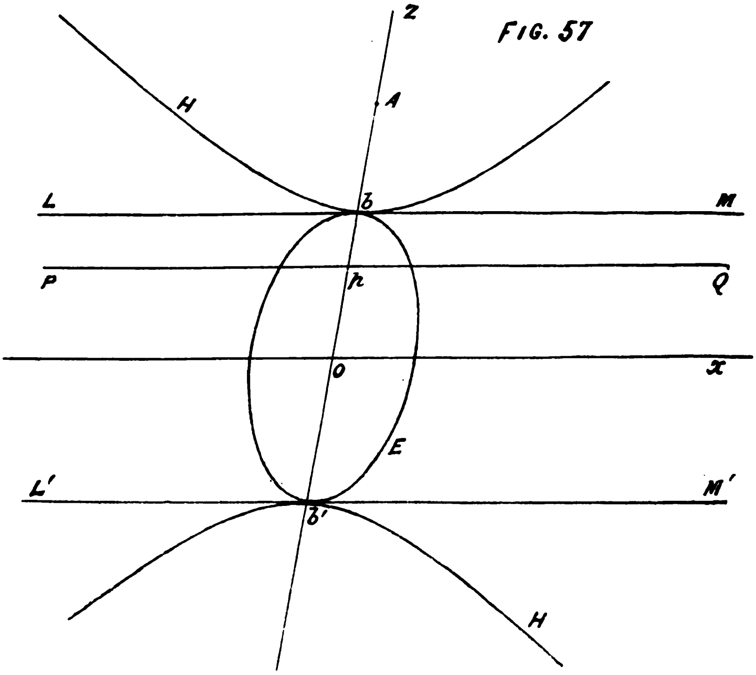


FIG. 24

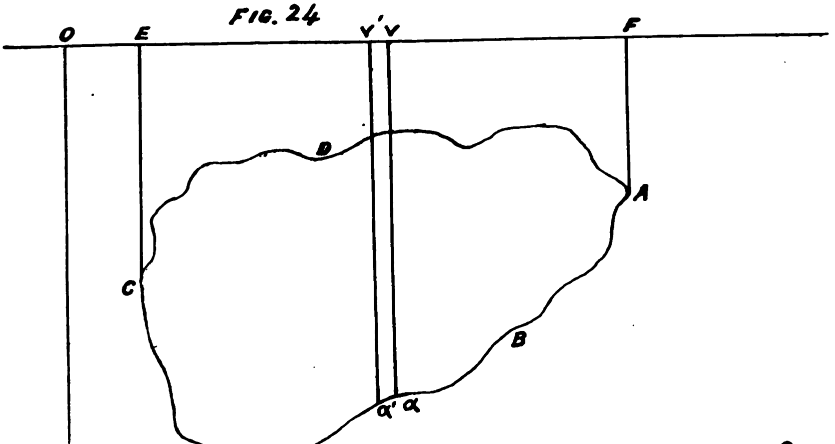


FIG. 25

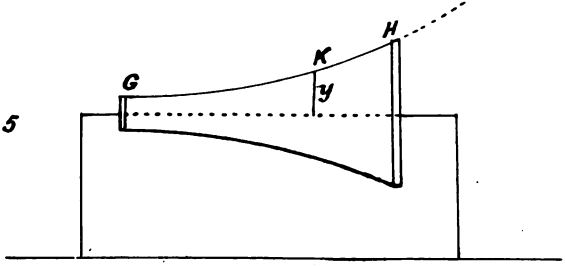


FIG. 26

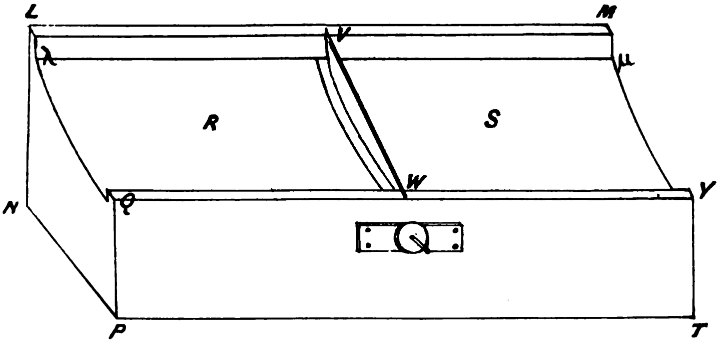


FIG. 27

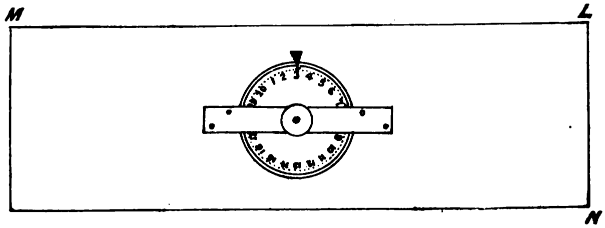
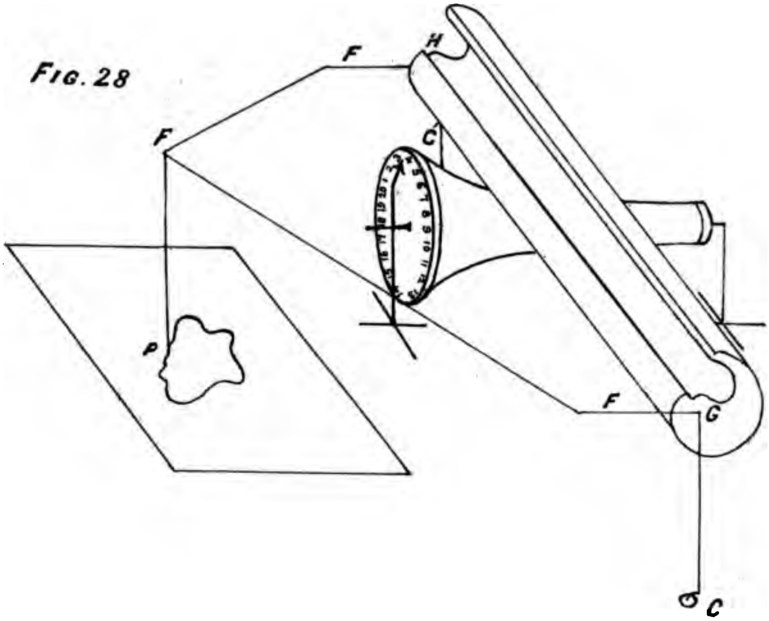


FIG. 28



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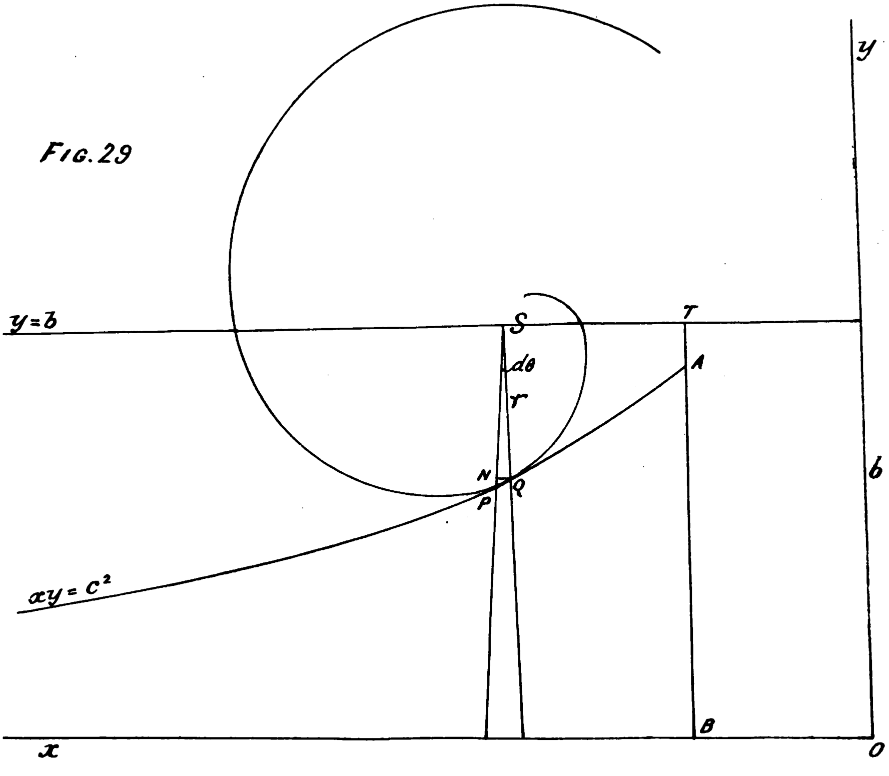


FIG. 30

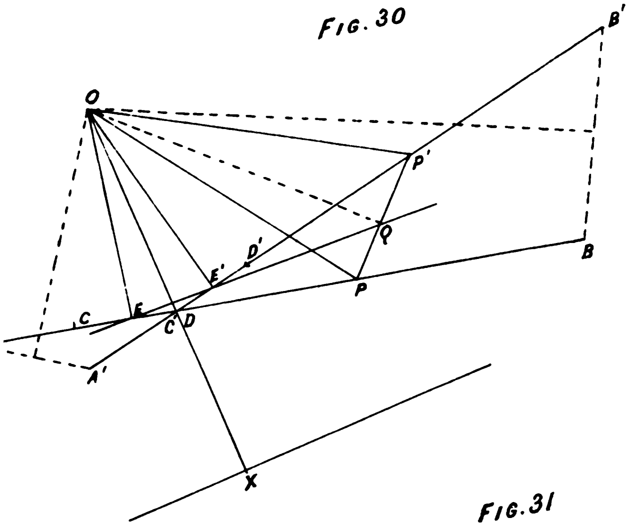


FIG. 31

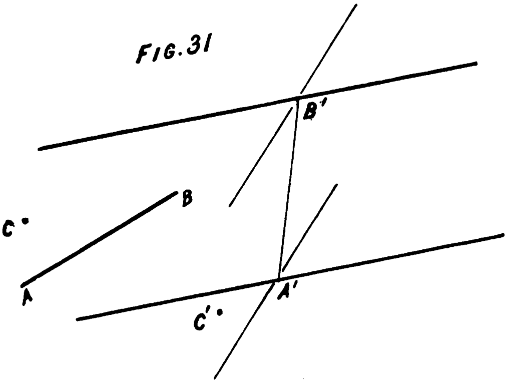


FIG. 32

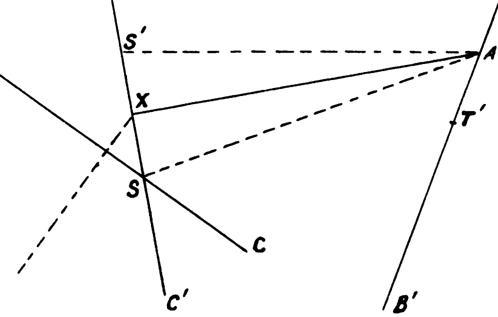


FIG. 33

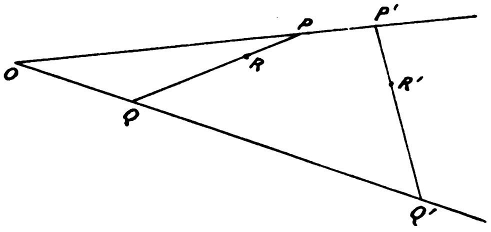


FIG. 34

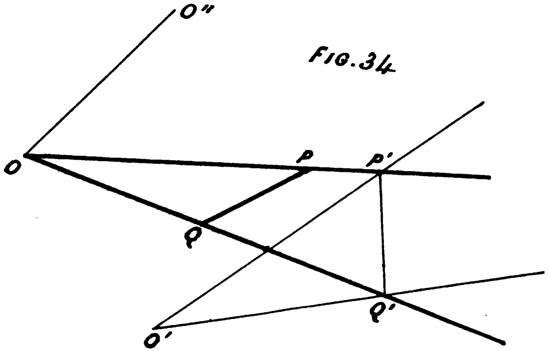


FIG. 35

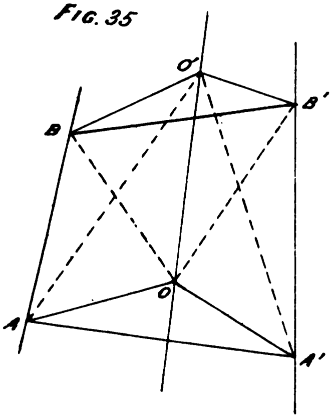


FIG. 36

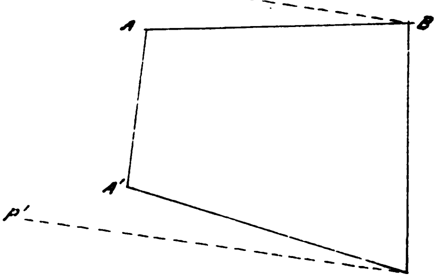
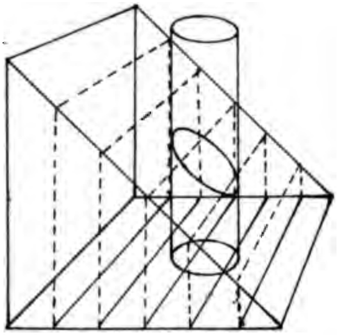


FIG. 37



Intersection of Plane and Cylinder.

FIG. 38

Projection of intersecting Cylinder and Cone.

A, axis of Cone.

C, axis of Cylinder.

P, point of orthogonal intersection of contours with Cylinder.

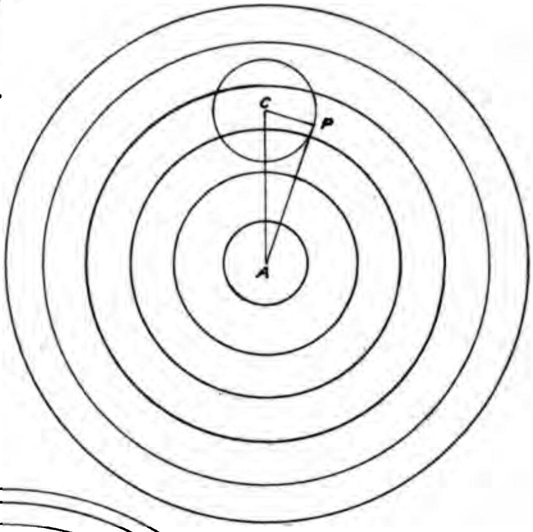
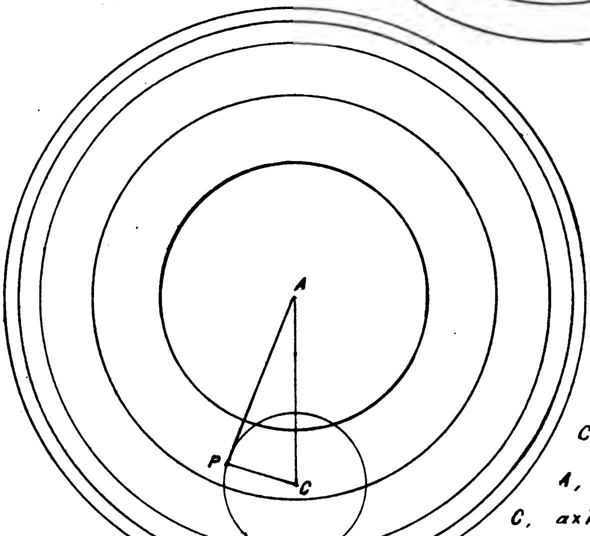


FIG. 39



Projection of intersecting Cylinder and Sphere

A, axis of Sphere.

C, axis of Cylinder.

P, point of orthogonal intersection of Contours with Cylinder.

FIG. 40

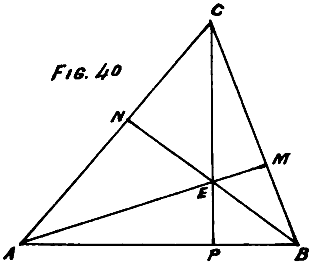


FIG. 41

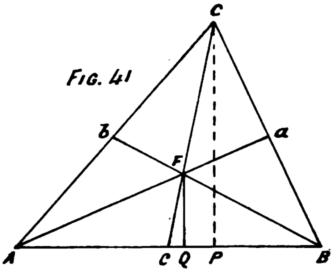


FIG. 42

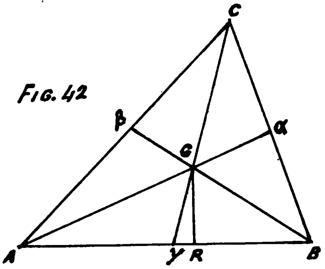


FIG. 43

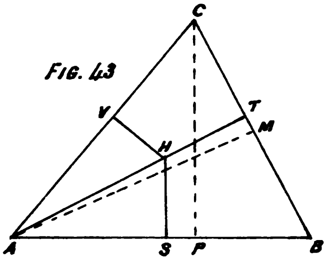


FIG. 44

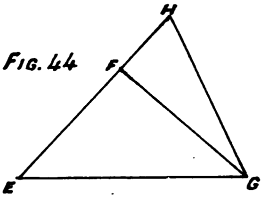


FIG. 45

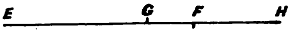


FIG. 46

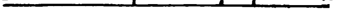


FIG. 47

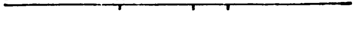


FIG. 48

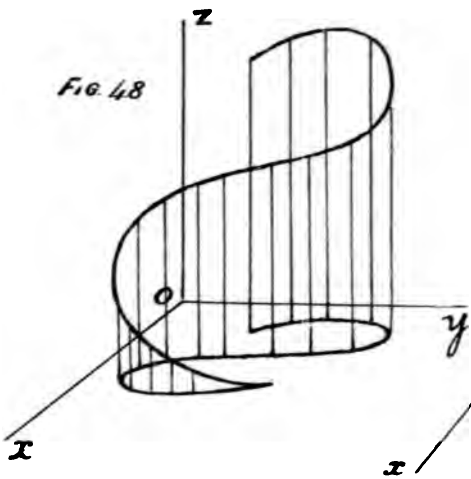


FIG. 49

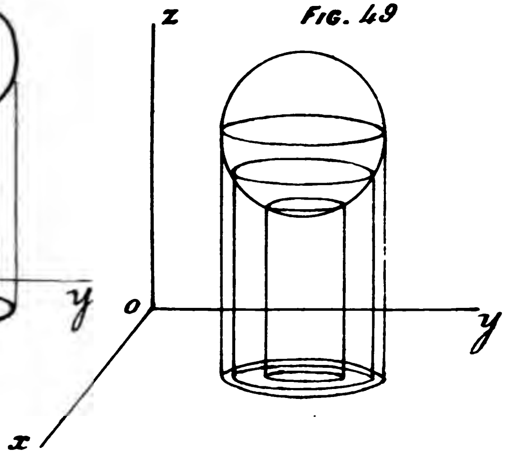


FIG. 50

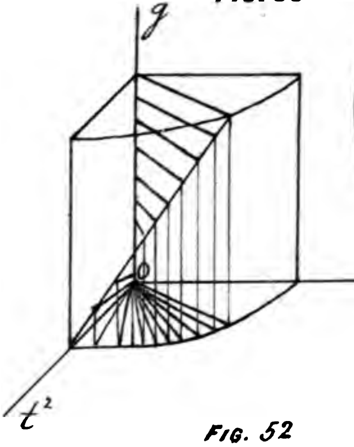


FIG. 51

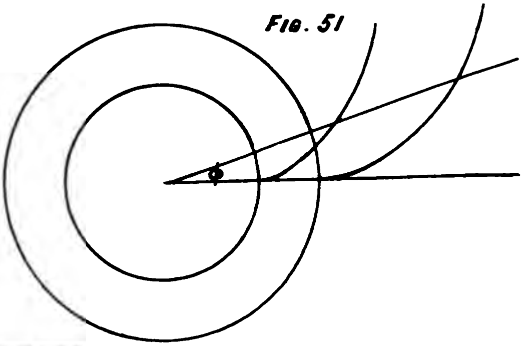


FIG. 52

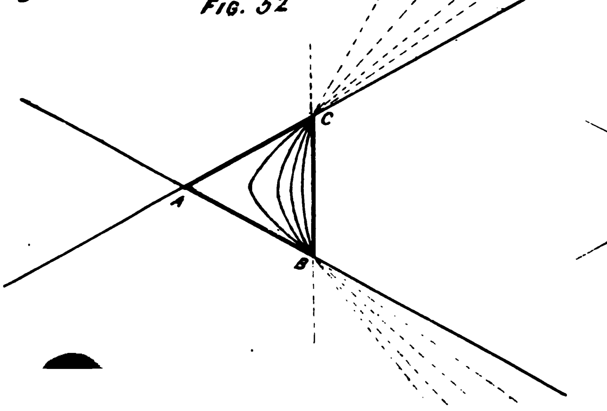


FIG. 53

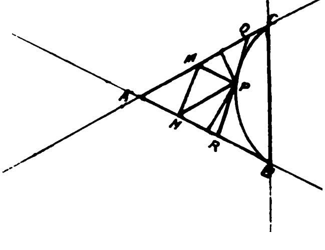


FIG. 54

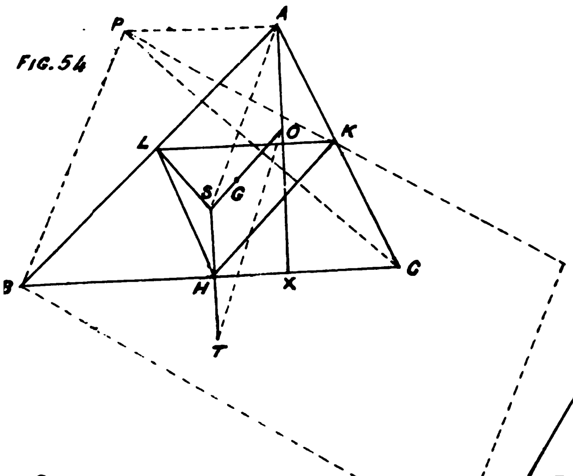


FIG. 55

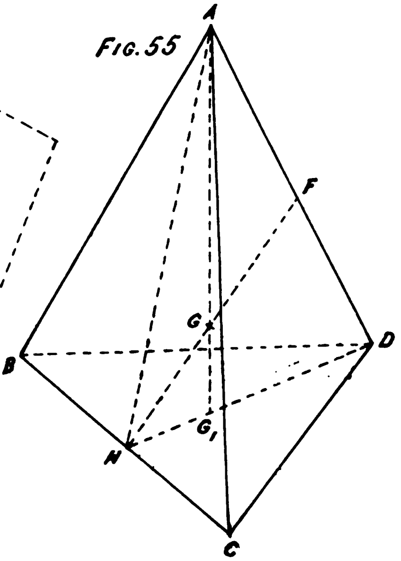


FIG. 56

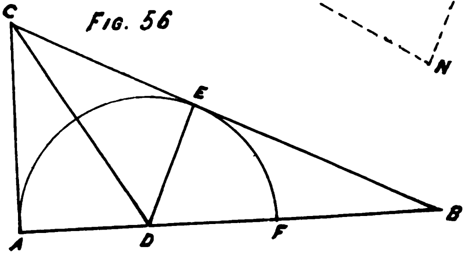
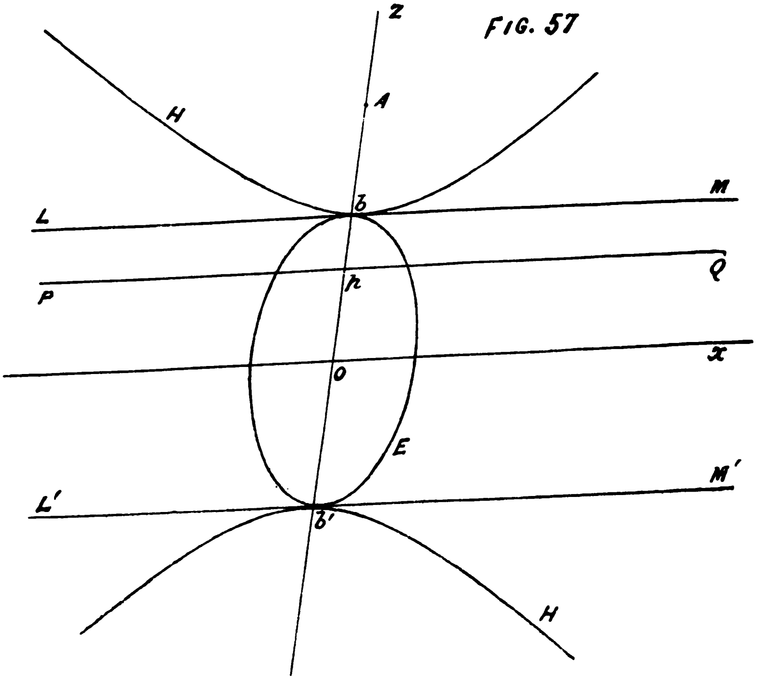


FIG. 57



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